

Past-paper walk-through

Finding Particular Solutions Using Initial Conditions and Separation of Variables ■

7.7

eXercise

Questions in this set

- 2026 Q3
- 2024 Q3
- 2023 Q3
- 2022 Q5
- 2021 Q6
- 2019 Q4
- 2018 Q6
- 2017 Q4
- 2016 Q4
- 2014 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- **//** separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 3

2026 ■ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

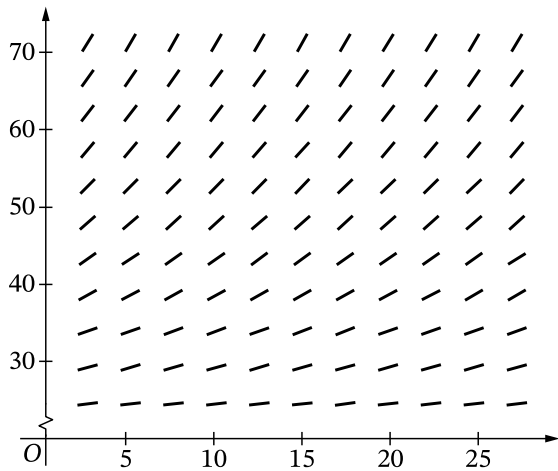
(a) Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.

[Slope field diagram: H -axis (vertical) from 30 to 70+ in increments of 10, t -axis (horizontal) from 0 to 25+ in increments of 5. At every grid point across the full range of t shown, the line segments have the same positive slope for a given horizontal row and the slopes appear to depend only on t (segments in each row are parallel and tilt

Question 3

more steeply positive at greater height H , but they do not appear to flatten toward zero as $H \rightarrow 20$ nor do the segments vary with t within a row as required) — segments are drawn as short parallel dashes tilting up-and-to-the-right throughout, uniformly across each horizontal band of H values, without change as t increases.]

Question 3



Question 3

2026 ■ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation

$\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(b) Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Question 3

2026 ■ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation

$\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(c) It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Question 3

2026 ■ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation

$\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(d) Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

Question 3

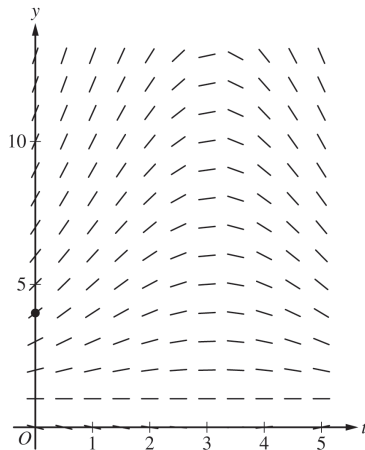
2024 ■ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1) \cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.

[Slope field graph; t -axis (horizontal) labeled 1 to 5, y -axis (vertical) labeled 5 and 10; a dot is plotted at $(0, 4)$; short line segments show the slope at each grid point — segments are steep and negatively-sloped (like "/") in the upper-right region, nearly flat in the lower region, and forward-sloped (like "/") in the upper-left region, consistent with the given differential equation.]

Question 3



Question 3

2024 ■ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1) \cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t , for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

Question 3

2024 ■ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1) \cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1) \cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

Solution 3

(a) Mark scheme

- The solution curve must pass through $(0, 4)$ and follow the given slope field: it rises (following the steep positive-then-shallowing slopes), reaches a relative maximum near $t = \pi \approx 3.14$ (height about 9.4, matching part (b)'s critical point), then decreases for $\pi < t < 5$, staying consistent with the marked segments and extending to at least $t = 4.5$ with no obvious conflicts with the drawn slope lines. 1 point awarded only if the curve passes through $(0, 4)$, extends to at least $t = 4.5$, and has no obvious conflicts with the given slope lines (only the portion inside the given field is considered).

Solution 3

(b) Mark scheme

- Since $H(t) > 1$ on $0 < t < 5$, $\frac{dH}{dt} = 0$ requires $\cos\left(\frac{t}{2}\right) = 0$, which gives $t = \pi$.
For $0 < t < \pi$, $\frac{dH}{dt} > 0$, and for $\pi < t < 5$, $\frac{dH}{dt} < 0$; therefore $t = \pi$ is the location of a relative maximum of H (the depth of seawater). 1 point for considering the sign of dH/dt (or equivalently $\cos(t/2)$), 1 point for identifying $t = \pi$ as the critical point, 1 point (only earnable with the first) for the correct classification 'relative maximum' with justification via a sign analysis of dH/dt on either side of $t = \pi$ (or an equivalent Second Derivative Test).

Solution 3

(c) Mark scheme

- Separate variables: $\frac{dH}{H-1} = \frac{1}{2} \cos\left(\frac{t}{2}\right) dt$. Integrate both sides:
 $\ln |H-1| = \sin\left(\frac{t}{2}\right) + C$. Apply $H(0) = 4$: $\ln |4-1| = \sin(0) + C \Rightarrow C = \ln 3$;
 since $H(0) = 4 > 1$, $|H-1| = H-1$, so $\ln(H-1) = \sin(t/2) + \ln 3$. Solve for H :
 $H-1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)}$, so $H(t) = 1 + 3e^{\sin(t/2)}$. 1 point for correct separation of variables, 1 point for one correct antiderivative (either side), 1 point for the second correct antiderivative, 1 point for including the constant of integration and correctly using the initial condition $H(0) = 4$ (eligible only if the separation point and at least one antiderivative point were earned), 1 point for the

Solution 3

fully solved final answer $H(t) = 1 + 3e^{\sin(t/2)}$ (eligible only if the first four points were earned; a response with no separation of variables earns 0 of 5).

Question 3

2023 ■ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t , where $M(t)$ is measured in degrees Celsius ($^{\circ}\text{C}$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation

$$\frac{dM}{dt} = \frac{1}{4}(40 - M).$$

At time $t = 0$, the temperature of the milk is 5°C . It can be shown that $M(t) < 40$ for all values of t .

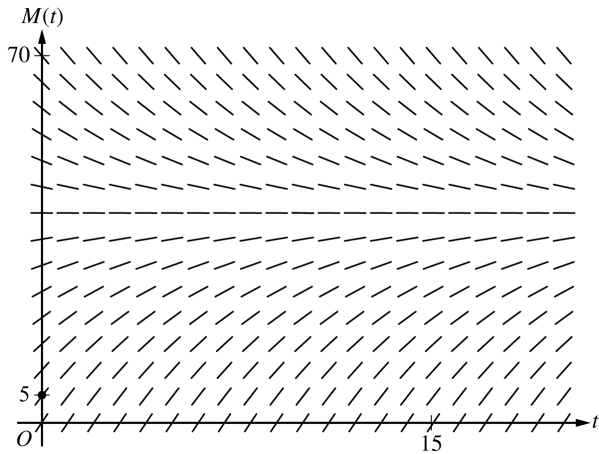
(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.

[Slope field diagram: axes labeled $M(t)$ (vertical, with gridlines at 5 and 70 marked) and t (horizontal, with a gridline at 15 marked), origin O . Short line segments are

Question 3

drawn at grid points across the region $0 \leq t \leq 15$ -ish, $0 \leq M(t) \leq 70$ -ish: segments are nearly horizontal (flat, slightly downward-sloping) near $M(t) = 70$, becoming steeper negative slope moving down toward $M(t) \approx 40$ (segments are horizontal/flat right at $M(t) = 40$), then the slopes become positive and increasingly steep as $M(t)$ decreases from 40 down toward $M(t) = 5$, with the steepest (near-vertical, slope ≈ 1) segments at the bottom of the field near $M(t) = 5$. The point $(0, 5)$ is marked with a star on the vertical axis.]

Question 3



Question 3

2023 ■ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t , where $M(t)$ is measured in degrees Celsius ($^{\circ}\text{C}$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C . It can be shown that $M(t) < 40$ for all values of t .

(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

Question 3

2023 ■ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t , where $M(t)$ is measured in degrees Celsius ($^{\circ}\text{C}$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C . It can be shown that $M(t) < 40$ for all values of t .

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M . Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

Question 3

2023 ■ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t , where $M(t)$ is measured in degrees Celsius ($^{\circ}\text{C}$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C . It can be shown that $M(t) < 40$ for all values of t .

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Solution 3

(a)

Mark scheme

- The solution curve through $(0,5)$ starts at $(0,5)$, increases and is concave down, and approaches (curving up toward, but always staying below) the flat slope segments at $M=40$ as t increases across the shown window; it must not cross any drawn slope-field segments (1 pt: curve passes through $(0,5)$, extends reasonably close to the left and right edges of the given rectangle, has no obvious conflicts with the slope lines, and lies entirely below $M=40$).

Solution 3

(b) Mark scheme

- $\left. \frac{dM}{dt} \right|_{t=0} = \frac{1}{4}(40 - 5) = \frac{35}{4}$ (1 pt: value of the derivative at $t=0$). Tangent line:
 $y = 5 + \frac{35}{4}(t - 0)$, so $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$; the temperature of the milk at $t=2$ minutes is approximately 22.5°C (1 pt: approximation using a tangent line through $(0,5)$ with slope $35/4$, correctly simplified).

Solution 3

(c) Mark scheme

- $\frac{d^2M}{dt^2} = -\frac{1}{4} \frac{dM}{dt} = -\frac{1}{4} \left(\frac{1}{4}(40 - M) \right) = -\frac{1}{16}(40 - M)$ (1 pt: correct second-derivative expression in terms of M , no subsequent simplification error).
Since $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down; therefore the tangent-line approximation from part (b) is an overestimate (1 pt: overestimate conclusion with a reason based on concavity/decreasing dM/dt).

Solution 3

(d) Mark scheme

- Separate variables: $\frac{dM}{40 - M} = \frac{1}{4} dt$ (1 pt: separates variables). Integrate:
 $\int \frac{dM}{40 - M} = \int \frac{1}{4} dt \Rightarrow -\ln|40 - M| = \frac{1}{4}t + C$ (1 pt: finds antiderivatives). Apply the initial condition $M(0)=5$: $-\ln|40 - 5| = 0 + C \Rightarrow C = -\ln 35$; since $M(t) < 40$, $|40 - M| = 40 - M$, giving $-\ln(40 - M) = \frac{1}{4}t - \ln 35$, i.e.
 $\ln(40 - M) = -\frac{1}{4}t + \ln 35$ (1 pt: constant of integration correctly found and initial condition used). Solve for M : $40 - M = 35e^{-t/4} \Rightarrow M(t) = 40 - 35e^{-t/4}$ (1 pt: solves for M — the final answer must be exactly this closed form, or equivalent).

Question 5

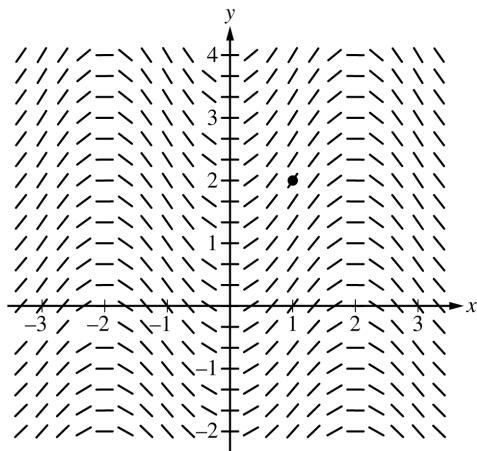
2022 ■ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) \sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.

[Graph: a slope field on the region $-3 \leq x \leq 3$, $-2 \leq y \leq 4$ (integer gridlines on both axes). Short line segments at each grid point show the local slope; segments in the left portion (roughly $x < 0$) slant like "/" (positive slope, forward slashes), and segments in the right portion (roughly $x > 0$) slant like "\" (negative slope, back slashes), with a marked point at $(1, 2)$ where the solution curve should be sketched.]

Question 5



Question 5

2022 ■ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) \sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

Question 5

2022 ■ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) \sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(c) It is known that $f''(x) > 0$ for $-1 \leq x \leq 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

Question 5

2022 ■ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) \sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) \sqrt{y+7}$$

with the initial condition $f(1) = 2$.

Solution 5

(a)

Mark scheme

- The solution curve must pass through $(1, 2)$, follow the given slope field, satisfy $f(x) > 0$ at every point on the curve, extend reasonably close to the left and right edges of the given square, and have any local maximum/minimum occur only where the slope field shows horizontal segments. This is a graphical sketch, not an algebraic answer; 1 point is earned for a curve satisfying these properties.

Solution 5

(b)

Mark scheme

- $\frac{dy}{dx}\bigg|_{(x,y)=(1,2)} = \frac{1}{2} \sin\left(\frac{\pi}{2}\right) \sqrt{2+7} = \frac{1}{2} \cdot 1 \cdot 3 = \frac{3}{2}$. Tangent line:
 $y = 2 + \frac{3}{2}(x - 1)$. Approximation: $f(0.8) \approx 2 + \frac{3}{2}(0.8 - 1) = 1.7$. Points: (1) correct tangent line equation (any equivalent form); (2) correct approximation $f(0.8) \approx 1.7$.

Solution 5

(c)

Mark scheme

- Because $f''(x) > 0$ on $-1 \leq x \leq 1$, f is concave up there, so the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ found in part (b) is an underestimate. 1 point for the correct answer with a reason that references $f''(x) > 0$, f' increasing, or f concave up.

Solution 5

(d) Mark scheme

■ $\frac{dy}{dx} = \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) \sqrt{y+7} \Rightarrow \int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2} \sin\left(\frac{\pi}{2}x\right) dx$. Integrating:

$$2\sqrt{y+7} = -\frac{1}{\pi} \cos\left(\frac{\pi}{2}x\right) + C. \text{ Using } f(1) = 2:$$

$$2\sqrt{2+7} = -\frac{1}{\pi} \cos\left(\frac{\pi}{2}\right) + C \Rightarrow 6 = 0 + C \Rightarrow C = 6. \text{ So}$$

$$\sqrt{y+7} = 3 - \frac{1}{2\pi} \cos\left(\frac{\pi}{2}x\right), \text{ and } y = \left(3 - \frac{1}{2\pi} \cos\left(\frac{\pi}{2}x\right)\right)^2 - 7. \text{ Points: (1)}$$

correct separation of variables; (2) one correct antiderivative; (3) the other correct antiderivative; (4) correct constant of integration found using the initial condition; (5) correctly solves for y .

Question 6

2021 ■ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

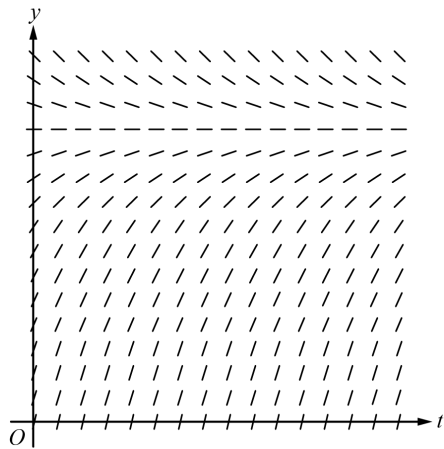
(a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.

[Slope field diagram, t -axis horizontal from 0 rightward, y -axis vertical; short line segments drawn at grid points showing slope direction: segments near the top of the grid slope downward (negative slope, since $y > 12$ there), a horizontal row of flat dashes at $y = 12$ (zero slope), and segments below slope upward (positive slope, since

Question 6

$y < 12$), with segments closer to $y = 0$ sloping steeply upward and flattening as y approaches 12 across each column.]

Question 6



Question 6

2021 ■ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(b) Using correct units, interpret the statement $\lim_{t \rightarrow \infty} A(t) = 12$ in the context of this problem.

Question 6

2021 ■ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ with initial condition $A(0) = 0$.

Question 6

2021 ■ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\frac{dy}{dt} = 3 - \frac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

Solution 6

(a) Mark scheme

- Sketch the solution curve to $\frac{dy}{dt} = \frac{12 - y}{3}$ through $(0, 0)$: the curve must start at $(0, 0)$, be generally increasing, be concave down throughout, and approach the horizontal asymptote $y = 12$ from below as t increases (following the slope-field segments, which are positive below $y = 12$, flat at $y = 12$, and negative above $y = 12$). Point 1: earned only if a single such curve is drawn passing through $(0, 0)$ with the correct increasing/concave-down/asymptotic shape; not earned if two or more solution curves are presented.

Solution 6

(b)

Mark scheme

- Over time, the amount of medication in the patient approaches 12 milligrams.
Point 1: the interpretation must include "medication in the patient," "approaches 12," and units (milligrams), or their equivalents.

Solution 6

(c) Mark scheme

- Separate variables: $\frac{dy}{dt} = \frac{12-y}{3} \Rightarrow \frac{dy}{12-y} = \frac{dt}{3}$. Integrate:

$$\int \frac{dy}{12-y} = \int \frac{dt}{3} \Rightarrow -\ln|12-y| = \frac{t}{3} + C. \text{ Solve for } y:$$

$$\ln|12-y| = -\frac{t}{3} - C \Rightarrow |12-y| = e^{-t/3-C} \Rightarrow y = 12 + Ke^{-t/3}. \text{ Apply } A(0) = 0:$$

$$0 = 12 + K \Rightarrow K = -12. \text{ Final: } y = A(t) = 12 - 12e^{-t/3}. \text{ Point 1: correct}$$

separation of variables ($dy/(12-y) = dt/3$; a "bad separation" like

$dy/(12-y) = 3 dt$ does not earn this point but the response stays eligible for points 2-3). Point 2: correct antiderivatives on both sides (absolute value bars not required). Point 3: correctly introduces and solves for the constant of integration

Solution 6

using the initial condition $A(0) = 0$ (earned consecutively after point 2). Point 4: solves explicitly for $y = A(t) = 12 - 12e^{-t/3}$ (needs points 1-3 in the standard path).

Solution 6

(d) Mark scheme

- $\frac{dy}{dt} = 3 - \frac{y}{t+2} \Rightarrow \frac{d^2y}{dt^2} = -\frac{\frac{dy}{dt}(t+2) - y}{(t+2)^2}$ (quotient rule). At $t = 1$:
 $B'(1) = 3 - \frac{B(1)}{3} = 3 - \frac{2.5}{3} = \frac{6.5}{3}$. Then
 $B''(1) = -\frac{B'(1) \cdot 3 - B(1)}{3^2} = -\frac{6.5 - 2.5}{9} = -\frac{4}{9} < 0$. The rate of change of the amount of medication in the second patient is DECREASING at $t = 1$ because $B''(1) < 0$ and d^2y/dt^2 is continuous on an interval containing $t = 1$. Point 1: correctly applies the quotient rule to $y/(t+2)$ (or product rule to $y(t+2)^{-1}$); errors differentiating the constant 3 or mishandling the sign forfeit this point. Point 2: states $B''(1) < 0$ with a correct value/sign — cannot be earned unless

Solution 6

the second derivative itself is correct. Point 3: consistent conclusion (decreasing) based on the declared value/sign of $B''(1)$; only needs an attempt at the quotient/product rule to be eligible.

Question 4

2019 ■ Q4

A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure below. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

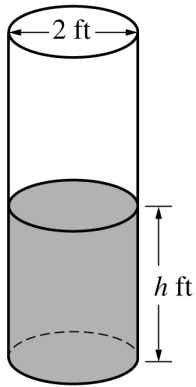
[Diagram of a cylindrical barrel with diameter labeled "2 ft" at the top opening; the barrel is partially filled with shaded (gray) water up to a level, with a vertical double-arrow labeled " h ft" marking the height of the water from the bottom of the barrel to the water's surface.]

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

Question 4

- (b)** When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.
- (c)** At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t .

Question 4



Solution 4

(a) Mark scheme

- $V = \pi r^2 h = \pi(1)^2 h = \pi h$ (radius fixed at 1 ft), so $\frac{dV}{dt} = \pi \frac{dh}{dt}$. At $h = 4$,
 $\frac{dh}{dt}\bigg|_{h=4} = -\frac{1}{10}\sqrt{4} = -\frac{1}{5}$. So $\frac{dV}{dt}\bigg|_{h=4} = \pi\left(-\frac{1}{5}\right) = -\frac{\pi}{5}$ cubic feet per second.
Points: 1 for $\frac{dV}{dt} = \pi \frac{dh}{dt}$, 1 for the correct answer with units ($-\pi/5$ cubic feet per second).

Solution 4

(b) Mark scheme

- $\frac{d^2h}{dt^2} = \frac{d}{dh} \left(-\frac{1}{10} \sqrt{h} \right) \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10} \sqrt{h} \right) = \frac{1}{200}$. Because $\frac{d^2h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height of the water is INCREASING when the height of the water is 3 feet. Points: 1 for $\frac{d}{dh} \left(-\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}}$, 1 for the chain-rule setup $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt}$, 1 for the answer (increasing) with explanation, since $d^2h/dt^2 = 1/200 > 0$.

Solution 4

(c) Mark scheme

- Separating variables: $\frac{dh}{\sqrt{h}} = -\frac{1}{10} dt$. Integrating:

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10} dt \Rightarrow 2\sqrt{h} = -\frac{1}{10}t + C. \text{ Using the initial condition } h(0) = 5:$$

$$2\sqrt{5} = -\frac{1}{10}(0) + C \Rightarrow C = 2\sqrt{5}. \text{ So } 2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}, \text{ and}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2. \text{ Points: 1 for separating variables, 1 for antiderivatives}$$

$(2\sqrt{h} \text{ and } -t/10)$, 1 for the constant of integration AND correctly using the initial condition $h(0) = 5$, 1 for the final expression $h(t) = (-t/20 + \sqrt{5})^2$. Note:

Solution 4

score 0/4 if no separation of variables; max 2/4 (pattern 1-1-0-0) if no constant of integration.

Question 6

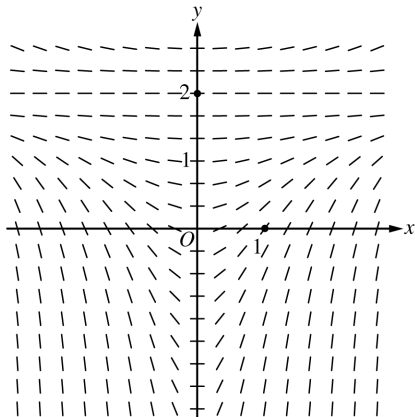
2018 ■ Q6

Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.

[Slope field for $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$ on axes with x -axis labeled O , 1 and y -axis labeled -1 , 1 , 2 . For $y > 2$ and $x < 0$, the segments slope downward (negative); for $y > 2$ and $x > 0$, the segments slope upward steeply (positive) except flattening to horizontal exactly at $y = 2$ for all x ; below $y = 2$ the segments are nearly horizontal near $y = 2$ and steepen (positive slope for $x > 0$, negative slope for $x < 0$) further from $y = 2$.]

Question 6



Question 6

2018 ■ Q6

Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

Question 6

2018 ■ Q6

Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

Solution 6

(a) Mark scheme

- Sketch two solution curves on the given slope field for $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$: one through $(0, 2)$ that follows the slope lines and stays essentially flat along $y = 2$ (since $y = 2$ is an equilibrium solution, slope 0 there) extending to the boundary of the field; and one through $(1, 0)$ that dips down following the field (roughly through the region near the origin) then curves back up toward $y = 2$ as x increases, also extending to the boundary. (1 pt: correct curve through $(0, 2)$; 1 pt: correct curve through $(1, 0)$. Both curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.)

Solution 6

(b)

Mark scheme

- $\left. \frac{dy}{dx} \right|_{(x,y)=(1,0)} = \frac{1}{3}(1)(0-2)^2 = \frac{4}{3}$. An equation for the line tangent to the graph of $y = f(x)$ at $x = 1$ is $y = \frac{4}{3}(x - 1)$. Using this tangent line, $f(0.7) \approx \frac{4}{3}(0.7 - 1) = -0.4$. (1 pt: correct equation of the tangent line; 1 pt: the approximation $f(0.7) \approx -0.4$.)

Solution 6

(c) Mark scheme

■ Separate variables: $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2 \Rightarrow \int \frac{dy}{(y-2)^2} = \int \frac{1}{3}x dx \Rightarrow \frac{-1}{y-2} = \frac{1}{6}x^2 + C.$

Using $f(1) = 0$: $\frac{1}{2} = \frac{1}{6} + C \Rightarrow C = \frac{1}{3}$. So $\frac{-1}{y-2} = \frac{1}{6}x^2 + \frac{1}{3} = \frac{x^2+2}{6}$, giving

$y = 2 - \frac{6}{x^2+2}$ (valid for $-\infty < x < \infty$). (1 pt: separates variables; 2 pts: correct antiderivatives on both sides; 1 pt: constant of integration found using the initial condition; 1 pt: solves explicitly for y . Note: 0/5 if no separation of variables; max 3/5 if no constant of integration is found/used.)

Question 4

2017 ■ Q4

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91°C at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

Question 4

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

Question 4

2016 ■ Q4

Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

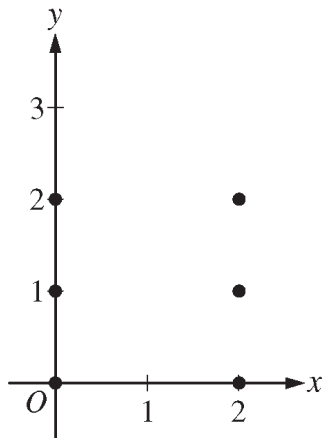
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

[Slope-field axes: x -axis labeled with tick marks at 1 and 2, y -axis labeled with tick marks at 1, 2, 3; the six indicated points (dots) are at $(0, 0)$, $(0, 1)$, $(0, 2)$, $(2, 0)$, $(2, 1)$, $(2, 2)$.]

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

Question 4



Solution 4

(a) Mark scheme

- Slope field at the six indicated points, using $dy/dx = y^2/(x-1)$: at $(0,0)$: slope = $0^2/(0-1)=0$; at $(2,0)$: slope = $0^2/(2-1)=0$ — these are the ZERO-slope points (horizontal segments). At $(0,1)$: slope = $1^2/(0-1)=-1$; at $(0,2)$: slope = $2^2/(0-1)=-4$; at $(2,1)$: slope = $1^2/(2-1)=1$; at $(2,2)$: slope = $2^2/(2-1)=4$ — these are the NONZERO-slope points, drawn as short segments with the corresponding sign and relative steepness. Points: 1 for correctly drawing zero slopes at $(0,0)$ and $(2,0)$, 1 for correctly drawing nonzero slopes (correct sign and relative steepness) at the other four points.

Solution 4

(b)

Mark scheme

■ $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = \frac{3^2}{2-1} = 9. \text{ Tangentline : } y = 9(x - 2) + 3. \text{ Approximation : } f(2.1) \approx 9(2.1 - 2) + 3 = 9(0.1) + 3 = 3.9. \text{ Points : } 1 \text{ for the correct tangent line equation, } 1 \text{ for the correct approximation } f(2.1) \approx 3.9.$

Solution 4

(c) Mark scheme

- Separate variables:

$$\frac{1}{y^2} dy = \frac{1}{x-1} dx. \text{ Integrate : } \int \frac{1}{y^2} dy = \int \frac{1}{x-1} dx \Rightarrow -\frac{1}{y} = \ln|x-1| + C.$$

Apply the initial condition $f(2) = 3$: $-\frac{1}{3} = \ln|2-1| + C = 0 + C \Rightarrow C = -\frac{1}{3}$. So $-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$.

Solve for y : $y = \frac{1}{\frac{1}{3} - \ln(x-1)}$ (valid for $1 < x < 1 + e^{1/3}$). Points :

1 for separating variables, 2 for correct antiderivatives (1 for each side), 1 for including the constant of integration, max 3/5 if no constant of integration is included; 0/5 if variables are not separated.

Question 6

2014 ■ Q6

Consider the differential equation $\frac{dy}{dx} = (3 - y) \cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

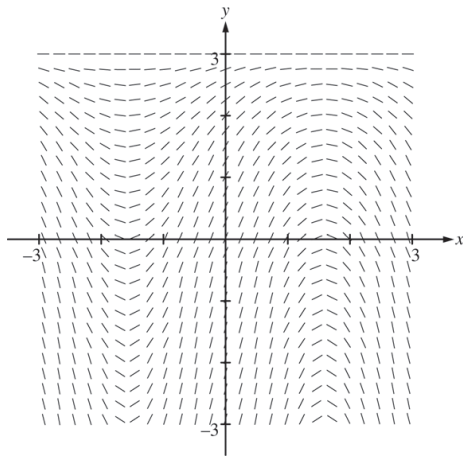
(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.

[Slope field diagram on the domain $-3 \leq x \leq 3$, $-3 \leq y \leq 3$, with tick marks at -3 , 1 , 2 (implied), 3 on both axes (visible ticks at -3 and 3 on the x -axis, at 3 and -3 on the y -axis, plus unlabelled tick marks at intermediate integer values). Short line segments show the slope at each grid point: segments are roughly horizontal (dashes) in the upper-left and upper-right regions (where y is large, slope near 0 since $3 - y$ is very negative but $\cos x$ oscillates - shown as near-flat dashes), segments tilt more

Question 6

steeply (steep near-vertical strokes) in the band around $y \approx 1$ to 2 near $x = 0$, and the pattern is symmetric in a banded way across the vertical line $x = 0$, repeating roughly periodically along the x -axis consistent with the $\cos x$ factor.]

Question 6



Question 6

2014 ■ Q6

Consider the differential equation $\frac{dy}{dx} = (3 - y) \cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

Question 6

2014 ■ Q6

Consider the differential equation $\frac{dy}{dx} = (3 - y) \cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

Solution 6

(a) Mark scheme

- Sketch the solution curve through $(0,1)$, following the slope-field segment directions: the curve should dip to a local minimum to the left of $x=0$ (matching the field's near-horizontal/negative segments there) and rise through $(0,1)$ toward a maximum further to the right, staying consistent with the plotted slope segments across $-3 \leq x \leq 3$ (matches the exact solution $y = 3 - 2e^{-\sin x}$, which has a minimum near $x = -\pi/2$ and a maximum near $x = \pi/2$). Points :
1 for a curve that passes through $(0,1)$ and is consistent with the given slope field's segment directions

Solution 6

(b)

Mark scheme

■ $\frac{dy}{dx}$

$$\left|_{(0,1)} = (3-1) \cos(0) = 2 \cdot 1 = 2. \text{ Tangent line: } y = 2x + 1 (\text{point-slope through } (0,1) \text{ with slope } 2). \text{ Using the tangent line, } f(0.2)$$

Solution 6

(c) Mark scheme

- Separate variables:

$$\frac{dy}{dx} = (3-y) \cos x \Rightarrow \int \frac{dy}{3-y} = \int \cos x \, dx \Rightarrow -\ln |3-y| = \sin x + C. \text{ Apply the initial condition } f(0) = 1: -\ln |3-1| = \sin(0) + C \Rightarrow -\ln 2 = C$$