Past-paper walk-through

Sketching Slope Fields ▪ 7.3

eXercise

Questions in this set

- 2024 Q3
- 2023 Q3
- 2022 Q5
- 2021 Q6
- 2018 Q6
- 2016 Q4
- 2015 Q4
- 2014 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

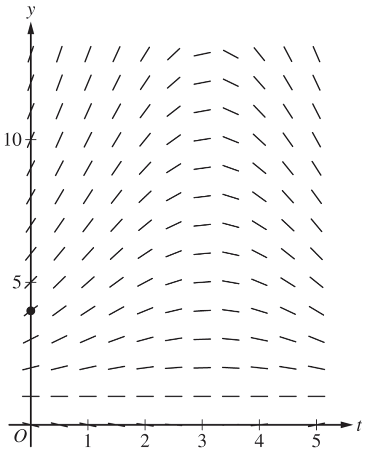
2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t = 0)$. It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.

[Slope field graph; t-axis (horizontal) labeled 1 to 5, y-axis (vertical) labeled 5 and 10; a dot is plotted at $(0, 4)$; short line segments show the slope at each grid point — segments are steep and negatively-sloped (like "") in the upper-right region, nearly flat in the lower region, and forward-sloped (like "/") in the upper-left region, consistent with the given differential equation.]

Question 3



Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t=0)$. It is known that $H(0)=4$.

(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t=0)$. It is known that $H(0)=4$.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0)=4$.

Solution 3

(a) Mark scheme

- The solution curve must pass through $(0, 4)$ and follow the given slope field: it rises (following the steep positive-then-shallowing slopes), reaches a relative maximum near $t = \pi \approx 3.14$ (height about 9.4, matching part (b)'s critical point), then decreases for $\pi < t < 5$, staying consistent with the marked segments and extending to at least $t = 4.5$ with no obvious conflicts with the drawn slope lines. 1 point awarded only if the curve passes through $(0, 4)$, extends to at least $t = 4.5$, and has no obvious conflicts with the given slope lines (only the portion inside the given field is considered).

Solution 3

(b) Mark scheme

- Since $H(t) > 1$ on $0 < t < 5$, $\dfrac{dH}{dt} = 0$ requires $\cos\left(\dfrac{t}{2}\right) = 0$, which gives $t = \pi$. For $0 < t < \pi$, $\dfrac{dH}{dt} > 0$, and for $\pi < t < 5$, $\dfrac{dH}{dt} < 0$; therefore $t = \pi$ is the location of a relative maximum of H (the depth of seawater). 1 point for considering the sign of dH/dt (or equivalently $\cos(t/2)$), 1 point for identifying $t = \pi$ as the critical point, 1 point (only earnable with the first) for the correct classification 'relative maximum' with justification via a sign analysis of dH/dt on either side of $t = \pi$ (or an equivalent Second Derivative Test).

Solution 3

(c) Mark scheme

- Separate variables: $\dfrac{dH}{H-1} = \dfrac{1}{2}\cos\left(\dfrac{t}{2}\right) dt$. Integrate both sides: $\ln|H-1| = \sin\left(\dfrac{t}{2}\right) + C$. Apply $H(0) = 4$: $\ln|4-1| = \sin(0) + C \Rightarrow C = \ln 3$; since $H(0) = 4 > 1$, $|H-1| = H-1$, so $\ln(H-1) = \sin(t/2) + \ln 3$. Solve for H: $H - 1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)}$, so $H(t) = 1 + 3e^{\sin(t/2)}$. 1 point for correct separation of variables, 1 point for one correct antiderivative (either side), 1 point for the second correct antiderivative, 1 point for including the constant of integration and correctly using the initial condition $H(0) = 4$ (eligible only if the separation point and at least one antiderivative point were earned), 1 point for the

Solution 3

fully solved final answer $H(t) = 1 + 3e^{\sin(t/2)}$ (eligible only if the first four points were earned; a response with no separation of variables earns 0 of 5).

Question 3

2023 ▪ Q3

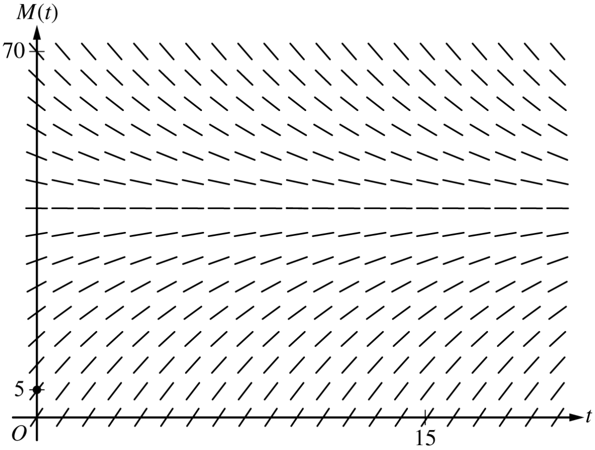
A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.
[Slope field diagram: axes labeled $M(t)$ (vertical, with gridlines at 5 and 70 marked) and t (horizontal, with a gridline at 15 marked), origin O. Short line segments are

Question 3

drawn at grid points across the region $0 \leq t \leq 15$-ish, $0 \leq M(t) \leq 70$-ish: segments are nearly horizontal (flat, slightly downward-sloping) near $M(t) = 70$, becoming steeper negative slope moving down toward $M(t) \approx 40$ (segments are horizontal/flat right at $M(t) = 40$), then the slopes become positive and increasingly steep as $M(t)$ decreases from 40 down toward $M(t) = 5$, with the steepest (near-vertical, slope ≈ 1) segments at the bottom of the field near $M(t) = 5$. The point $(0, 5)$ is marked with a star on the vertical axis.]

Question 3



Question 3

2023 ▪ Q3

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(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

Question 3

2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(c) Write an expression for $\dfrac{d^2M}{dt^2}$ in terms of M. Use $\dfrac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

Question 3

2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Solution 3

(a)

Mark scheme

- The solution curve through (0,5) starts at (0,5), increases and is concave down, and approaches (curving up toward, but always staying below) the flat slope segments at $M=40$ as t increases across the shown window; it must not cross any drawn slope-field segments (1 pt: curve passes through (0,5), extends reasonably close to the left and right edges of the given rectangle, has no obvious conflicts with the slope lines, and lies entirely below $M=40$).

Solution 3

(b) Mark scheme

- $\left.\dfrac{dM}{dt}\right|_{t=0} = \dfrac{1}{4}(40-5) = \dfrac{35}{4}$ (1 pt: value of the derivative at t=0). Tangent line: $y = 5 + \dfrac{35}{4}(t-0)$, so $M(2) \approx 5 + \dfrac{35}{4} \cdot 2 = 22.5$; the temperature of the milk at t=2 minutes is approximately 22.5°C (1 pt: approximation using a tangent line through (0,5) with slope 35/4, correctly simplified).

Solution 3

(c) Mark scheme

- $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right) = -\frac{1}{16}(40 - M)$ (1 pt: correct second-derivative expression in terms of M, no subsequent simplification error). Since $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down; therefore the tangent-line approximation from part (b) is an overestimate (1 pt: overestimate conclusion with a reason based on concavity/decreasing dM/dt).

Solution 3

(d) Mark scheme

- Separate variables: $\dfrac{dM}{40-M} = \dfrac{1}{4}\,dt$ (1 pt: separates variables). Integrate: $\displaystyle\int \frac{dM}{40-M} = \int \frac{1}{4}\,dt \Rightarrow -\ln|40-M| = \frac{1}{4}t + C$ (1 pt: finds antiderivatives). Apply the initial condition M(0)=5: $-\ln|40-5| = 0 + C \Rightarrow C = -\ln 35$; since M(t)<40, $|40-M| = 40-M$, giving $-\ln(40-M) = \dfrac{1}{4}t - \ln 35$, i.e. $\ln(40-M) = -\dfrac{1}{4}t + \ln 35$ (1 pt: constant of integration correctly found and initial condition used). Solve for M: $40 - M = 35e^{-t/4} \Rightarrow M(t) = 40 - 35e^{-t/4}$ (1 pt: solves for M — the final answer must be exactly this closed form, or equivalent).

Question 5

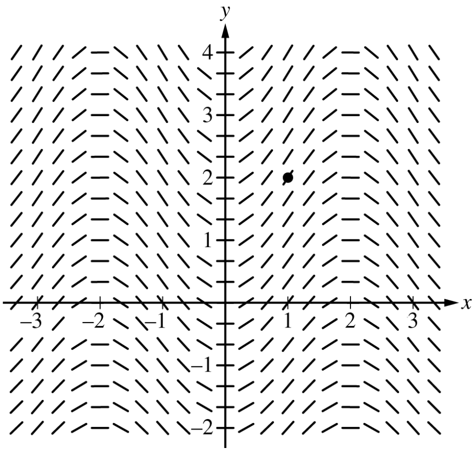
2022 ▪ Q5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.

[Graph: a slope field on the region $-3 \leq x \leq 3$, $-2 \leq y \leq 4$ (integer gridlines on both axes). Short line segments at each grid point show the local slope; segments in the left portion (roughly $x < 0$) slant like "/" (positive slope, forward slashes), and segments in the right portion (roughly $x > 0$) slant like "(negative slope, back slashes), with a marked point at $(1, 2)$ where the solution curve should be sketched.]

Question 5



Question 5

2022 ▪ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

Question 5

2022 ▪ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(c) It is known that $f''(x) > 0$ for $-1 \leq x \leq 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

Question 5

2022 ▪ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$$

with the initial condition $f(1) = 2$.

Solution 5

(a)

Mark scheme

- The solution curve must pass through $(1, 2)$, follow the given slope field, satisfy $f(x) > 0$ at every point on the curve, extend reasonably close to the left and right edges of the given square, and have any local maximum/minimum occur only where the slope field shows horizontal segments. This is a graphical sketch, not an algebraic answer; 1 point is earned for a curve satisfying these properties.

Solution 5

(b)

Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,2)} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}\right)\sqrt{2+7} = \dfrac{1}{2}\cdot 1\cdot 3 = \dfrac{3}{2}$. Tangent line: $y = 2 + \dfrac{3}{2}(x-1)$. Approximation: $f(0.8) \approx 2 + \dfrac{3}{2}(0.8-1) = 1.7$. Points: (1) correct tangent line equation (any equivalent form); (2) correct approximation $f(0.8) \approx 1.7$.

Solution 5

(c)

Mark scheme

- Because $f''(x) > 0$ on $-1 \leq x \leq 1$, f is concave up there, so the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ found in part (b) is an underestimate. 1 point for the correct answer with a reason that references $f''(x) > 0$, f' increasing, or f concave up.

Solution 5

(d) Mark scheme

- $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7} \Rightarrow \displaystyle\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right) dx$. Integrating: $2\sqrt{y+7} = -\dfrac{1}{\pi}\cos\left(\dfrac{\pi}{2}x\right) + C$. Using $f(1) = 2$: $2\sqrt{2+7} = -\dfrac{1}{\pi}\cos\left(\dfrac{\pi}{2}\right) + C \Rightarrow 6 = 0 + C \Rightarrow C = 6$. So $\sqrt{y+7} = 3 - \dfrac{1}{2\pi}\cos\left(\dfrac{\pi}{2}x\right)$, and $y = \left(3 - \dfrac{1}{2\pi}\cos\left(\dfrac{\pi}{2}x\right)\right)^2 - 7$. Points: (1) correct separation of variables; (2) one correct antiderivative; (3) the other correct antiderivative; (4) correct constant of integration found using the initial condition; (5) correctly solves for y.

Question 6

2021 ▪ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

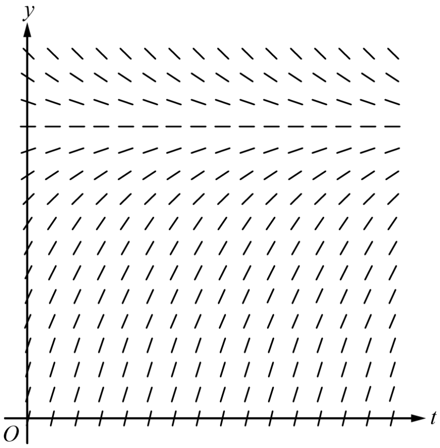
(a) A portion of the slope field for the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.

[Slope field diagram, t-axis horizontal from 0 rightward, y-axis vertical; short line segments drawn at grid points showing slope direction: segments near the top of the grid slope downward (negative slope, since $y > 12$ there), a horizontal row of flat dashes at $y = 12$ (zero slope), and segments below slope upward (positive slope, since

Question 6

$y < 12$), with segments closer to $y = 0$ sloping steeply upward and flattening as y approaches 12 across each column.]

Question 6



Question 6

2021 ▪ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(b) Using correct units, interpret the statement $\lim_{t \to \infty} A(t) = 12$ in the context of this problem.

Question 6

2021 ▪ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$ with initial condition $A(0) = 0$.

Question 6

2021 ▪ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\frac{dy}{dt} = 3 - \frac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

Solution 6

(a) Mark scheme

- Sketch the solution curve to $\frac{dy}{dt} = \frac{12-y}{3}$ through $(0,0)$: the curve must start at $(0,0)$, be generally increasing, be concave down throughout, and approach the horizontal asymptote $y = 12$ from below as t increases (following the slope-field segments, which are positive below $y = 12$, flat at $y = 12$, and negative above $y = 12$). Point 1: earned only if a single such curve is drawn passing through $(0,0)$ with the correct increasing/concave-down/asymptotic shape; not earned if two or more solution curves are presented.

Solution 6

(b)

Mark scheme

- Over time, the amount of medication in the patient approaches 12 milligrams. Point 1: the interpretation must include "medication in the patient," "approaches 12," and units (milligrams), or their equivalents.

Solution 6

(c) Mark scheme

- Separate variables: $\frac{dy}{dt} = \frac{12-y}{3} \Rightarrow \frac{dy}{12-y} = \frac{dt}{3}$. Integrate: $\int \frac{dy}{12-y} = \int \frac{dt}{3} \Rightarrow -\ln|12-y| = \frac{t}{3} + C$. Solve for y: $\ln|12-y| = -\frac{t}{3} - C \Rightarrow |12-y| = e^{-t/3-C} \Rightarrow y = 12 + Ke^{-t/3}$. Apply $A(0) = 0$: $0 = 12 + K \Rightarrow K = -12$. Final: $y = A(t) = 12 - 12e^{-t/3}$. Point 1: correct separation of variables ($dy/(12-y) = dt/3$; a "bad separation" like $dy/(12-y) = 3\,dt$ does not earn this point but the response stays eligible for points 2-3). Point 2: correct antiderivatives on both sides (absolute value bars not required). Point 3: correctly introduces and solves for the constant of integration

Solution 6

using the initial condition $A(0) = 0$ (earned consecutively after point 2). Point 4: solves explicitly for $y = A(t) = 12 - 12e^{-t/3}$ (needs points 1-3 in the standard path).

Solution 6

(d) Mark scheme

- $\dfrac{dy}{dt} = 3 - \dfrac{y}{t+2} \Rightarrow \dfrac{d^2y}{dt^2} = -\dfrac{\frac{dy}{dt}(t+2) - y}{(t+2)^2}$ (quotient rule). At $t = 1$: $B'(1) = 3 - \dfrac{B(1)}{3} = 3 - \dfrac{2.5}{3} = \dfrac{6.5}{3}$. Then $B''(1) = -\dfrac{B'(1) \cdot 3 - B(1)}{3^2} = -\dfrac{6.5 - 2.5}{9} = -\dfrac{4}{9} < 0$. The rate of change of the amount of medication in the second patient is DECREASING at $t = 1$ because $B''(1) < 0$ and d^2y/dt^2 is continuous on an interval containing $t = 1$. Point 1: correctly applies the quotient rule to $y/(t+2)$ (or product rule to $y(t+2)^{-1}$); errors differentiating the constant 3 or mishandling the sign forfeit this point. Point 2: states $B''(1) < 0$ with a correct value/sign — cannot be earned unless

Solution 6

the second derivative itself is correct. Point 3: consistent conclusion (decreasing) based on the declared value/sign of $B''(1)$; only needs an attempt at the quotient/product rule to be eligible.

Question 6

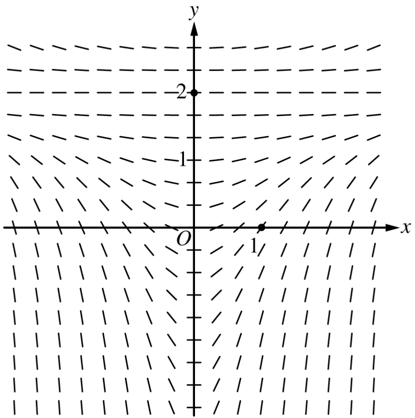
2018 ▪ Q6

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.

[Slope field for $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$ on axes with x-axis labeled O, 1 and y-axis labeled -1, 1, 2. For $y > 2$ and $x < 0$, the segments slope downward (negative); for $y > 2$ and $x > 0$, the segments slope upward steeply (positive) except flattening to horizontal exactly at $y = 2$ for all x; below $y = 2$ the segments are nearly horizontal near $y = 2$ and steepen (positive slope for $x > 0$, negative slope for $x < 0$) further from $y = 2$.]

Question 6



Question 6

2018 ▪ Q6

Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

Question 6

2018 ▪ Q6

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{3}x(y-2)^2$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

Solution 6

(a) Mark scheme

- Sketch two solution curves on the given slope field for $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$: one through $(0,2)$ that follows the slope lines and stays essentially flat along $y=2$ (since $y=2$ is an equilibrium solution, slope 0 there) extending to the boundary of the field; and one through $(1,0)$ that dips down following the field (roughly through the region near the origin) then curves back up toward $y=2$ as x increases, also extending to the boundary. (1 pt: correct curve through $(0,2)$; 1 pt: correct curve through $(1,0)$. Both curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.)

Solution 6

(b)

Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,0)} = \frac{1}{3}(1)(0-2)^2 = \frac{4}{3}$. An equation for the line tangent to the graph of $y = f(x)$ at $x = 1$ is $y = \frac{4}{3}(x-1)$. Using this tangent line, $f(0.7) \approx \frac{4}{3}(0.7-1) = -0.4$. (1 pt: correct equation of the tangent line; 1 pt: the approximation $f(0.7) \approx -0.4$.)

Solution 6

(c) Mark scheme

- Separate variables: $\dfrac{dy}{dx} = \frac{1}{3}x(y-2)^2 \Rightarrow \int \dfrac{dy}{(y-2)^2} = \int \frac{1}{3}x\,dx \Rightarrow \dfrac{-1}{y-2} = \frac{1}{6}x^2 + C$. Using $f(1) = 0$: $\frac{1}{2} = \frac{1}{6} + C \Rightarrow C = \frac{1}{3}$. So $\dfrac{-1}{y-2} = \frac{1}{6}x^2 + \frac{1}{3} = \dfrac{x^2+2}{6}$, giving $y = 2 - \dfrac{6}{x^2+2}$ (valid for $-\infty < x < \infty$). (1 pt: separates variables; 2 pts: correct antiderivatives on both sides; 1 pt: constant of integration found using the initial condition; 1 pt: solves explicitly for y. Note: 0/5 if no separation of variables; max 3/5 if no constant of integration is found/used.)

Question 4

2016 ▪ Q4

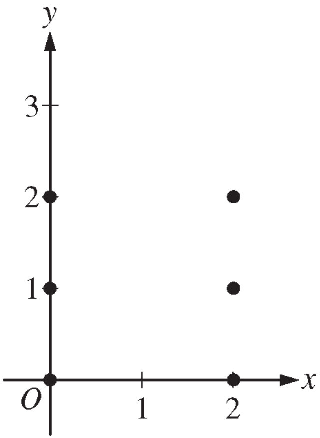
Consider the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
[Slope-field axes: x-axis labeled with tick marks at 1 and 2, y-axis labeled with tick marks at 1, 2, 3; the six indicated points (dots) are at $(0,0)$, $(0,1)$, $(0,2)$, $(2,0)$, $(2,1)$, $(2,2)$.]

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

Question 4



Solution 4

(a) Mark scheme

- Slope field at the six indicated points, using dy/dx = y^2/(x-1): at (0,0): slope = 0^2/(0-1)=0; at (2,0): slope = 0^2/(2-1)=0 — these are the ZERO-slope points (horizontal segments). At (0,1): slope = 1^2/(0-1)=-1; at (0,2): slope = 2^2/(0-1)=-4; at (2,1): slope = 1^2/(2-1)=1; at (2,2): slope = 2^2/(2-1)=4 — these are the NONZERO-slope points, drawn as short segments with the corresponding sign and relative steepness. Points: 1 for correctly drawing zero slopes at (0,0) and (2,0), 1 for correctly drawing nonzero slopes (correct sign and relative steepness) at the other four points.

Solution 4

(b)

Mark scheme

- $\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} = \frac{3^2}{2-1} = 9. Tangentline : y = 9(x-2)+3. Approximation : f(2.1) \approx 9(2.1-2)+3 = 9(0.1)+3 = 3.9. Points : 1 forthecorrecttangentlineequation, 1 forthecorrectapproximationf(2.1) \approx 3.9.$

Solution 4

(c) Mark scheme

- Separate variables:
$\frac{1}{y^2}\,dy = \frac{1}{x-1}\,dx.Integrate: \int \frac{1}{y^2}\,dy = \int \frac{1}{x-1}\,dx \Rightarrow -\frac{1}{y} = \ln|x-1| +$
$C.Applytheinitialconditionf(2) = 3: -\frac{1}{3} = \ln|2-1| + C = 0 + C \Rightarrow C = -\frac{1}{3}.So - \frac{1}{y} =$
$\ln|x-1| - \frac{1}{3}.Solvefory: y = \frac{1}{\frac{1}{3} - \ln(x-1)}(validfor1 < x < 1 + e^{1/3}).Points:$
$1forseparatingvariables, 2forcorrectantiderivatives(1foreachside), 1forincludingtheconstantofint$
$max3/5ifnoconstantofintegrationisincluded; 0/5ifvariablesarenotseparated.$

Question 4

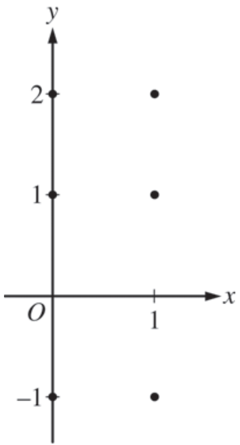
2015 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

[Slope-field axes labeled y (vertical) and x (horizontal), origin O. Six points are marked with dots, at coordinates $(1, 2)$, $(1, 1)$, $(1, -1)$, and their positions relative to axis tick marks at $y = 2$, $y = 1$, $y = -1$ on the y-axis and $x = 1$ on the x-axis. (The six indicated points are at $x = 1$ paired with $y = 2, 1, -1$, and at $x = 2$ paired with $y = 2, 1, -1$, based on the two columns of dots shown above $y = 2$, $y = 1$, and $y = -1$.)]

Question 4



Question 4

2015 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = 2x - y$.

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

Question 4

2015 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = 2x - y$.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

Question 4

2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

Solution 4

(a)

Mark scheme

- Slope field at the six indicated points using $dy/dx = 2x - y$. At $x = 0$: (0,2) slope $= -2$, (0,1) slope $= -1$, (0,−1) slope $= 1$. At $x = 1$: (1,2) slope $= 0$, (1,1) slope $= 1$, (1,−1) slope $= 3$. Points: 1 for correct slope segments drawn at the three points where $x = 0$, 1 for correct slope segments drawn at the three points where $x = 1$. (This is a drawing task — grade by checking the drawn segment directions/steepness at each of the six given points match these computed slopes, not a numeric or algebraic answer.)

Solution 4

(b)

Mark scheme

- $d^2y/dx^2 = 2 - dy/dx = 2 - (2x - y) = 2 - 2x + y$. In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$ (each term is positive). Therefore all solution curves are concave up in Quadrant II. Points: 1 for the correct expression $d^2y/dx^2 = 2 - 2x + y$, 1 for the correct conclusion (concave up) with a valid reason (sign argument in Quadrant II).

Solution 4

(c)

Mark scheme

- dy/dx at $(x, y) = (2, 3)$ is $2(2) - 3 = 1 \neq 0$. Therefore f has neither a relative minimum nor a relative maximum at $x = 2$, since the derivative is nonzero there. Points: 1 for considering dy/dx at $(2, 3)$, 1 for the correct conclusion (neither min nor max) with justification ($dy/dx \neq 0$ at that point).

Solution 4

(d)

Mark scheme

- If $y = mx + b$, then $dy/dx = d/dx(mx + b) = m$. Substituting into $2x - y = m$ gives $2x - (mx + b) = m$, i.e. $(2 - m)x - (m + b) = 0$ for all x, so $2 - m = 0 \quad m = 2$, and $-(m + b) = 0 \quad b = -m = -2$. Therefore $m = 2$ and $b = -2$. Points: 1 for $d/dx(mx + b) = m$, 1 for the equation $2x - y = m$ (substituting $y = mx + b$ into the differential equation, or equivalent), 1 for the correct answer $m = 2$, $b = -2$.

Question 6

2014 ▪ Q6

Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

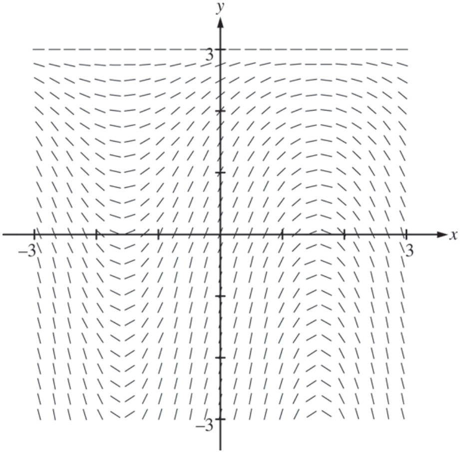
(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.

[Slope field diagram on the domain $-3 \leq x \leq 3$, $-3 \leq y \leq 3$, with tick marks at -3, 1, 2(implied), 3 on both axes (visible ticks at -3 and 3 on the x-axis, at 3 and -3 on the y-axis, plus unlabelled tick marks at intermediate integer values). Short line segments show the slope at each grid point: segments are roughly horizontal (dashes) in the upper-left and upper-right regions (where y is large, slope near 0 since $3 - y$ is very negative but cos x oscillates - shown as near-flat dashes), segments tilt more

Question 6

steeply (steep near-vertical strokes) in the band around $y \approx 1$ to 2 near $x = 0$, and the pattern is symmetric in a banded way across the vertical line $x = 0$, repeating roughly periodically along the x-axis consistent with the $\cos x$ factor.]

Question 6



Question 6

2014 ▪ Q6

Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

Question 6

2014 ▪ Q6

Consider the differential equation $\dfrac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

Solution 6

(a) Mark scheme

- Sketch the solution curve through (0,1), following the slope-field segment directions: the curve should dip to a local minimum to the left of x=0 (matching the field's near-horizontal/negative segments there) and rise through (0,1) toward a maximum further to the right, staying consistent with the plotted slope segments across -3$\leq x \leq 3 (matchestheexactsolutiony = 3 - 2e^{-\sin x}, whichhasaminimumnearx = -\pi/2 andamaximumnearx = \pi/2). Points : 1 foracurvethatpassesthrough (0,1) andisconsistentwiththegivenslopefield's segmentdirectionsthr

Solution 6

(b)

Mark scheme

- $dy \overline{\qquad} \; dx\Big|_{(0,1)} = (3-1)\cos(0) = 2\cdot 1 = 2. Tangentline{:} y = 2x + 1 (point - slopethrough(0,1) withslope 2). Usingthetangentline, f(0.2$

Solution 6

(c) Mark scheme

- Separate variables:

$$\frac{\text{dy}}{dx = (3-y)\cos x \Rightarrow \int \frac{dy}{3-y} = \int \cos x\, dx \Rightarrow -\ln|3-y| = \sin x + C. Applytheinitialcondition f(0)=1: -\ln|3-1| = \sin(0) + C \Rightarrow -\ln}$$