Past-paper walk-through

Determining Intervals on Which a Function Is Increasing or Decreasing ▪ 5.3

eXercise

Questions in this set

- 2026 Q4
- 2023 Q5
- 2022 Q1
- 2022 Q2
- 2022 Q3
- 2018 Q3
- 2017 Q2
- 2017 Q3
- 2015 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 4

2026 ▪ Q4

Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f', the derivative of f, is shown.
[Graph of f': x-axis from -4 to 4, y-axis from -1 to 5. The curve passes through labeled points $(-4, -0.5)$, a local minimum at $(-3, -1)$, rises through the x-axis between $x = -2$ and $x = -1$, continues rising to a local maximum at $(1, 3)$, then decreases through $(2, 1.5)$, crosses the x-axis at $x = 3$ (touching down to a minimum of 0 at $x = 3$), then rises sharply to $(4, 5)$.]

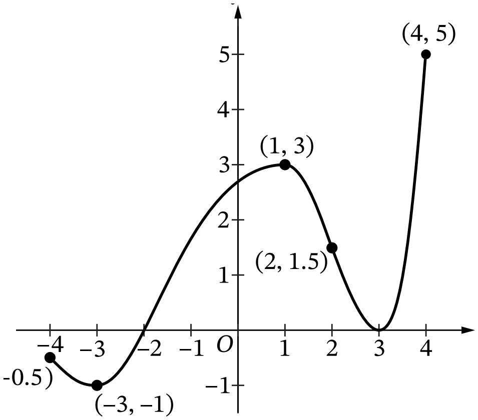
(a) For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

(b) Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

Question 4

(c) For $-4 \leq x \leq 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.
(d) For $-4 \leq x \leq 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

Question 4



Graph of f'

Question 5

2023 ▪ Q5

The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

(a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

(b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.

(c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

Question 5

(d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$? Justify your answer.

Question 5

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

Solution 5

(a)

Mark scheme

- $h'(x) = f'(g(x)) \cdot g'(x)$ (1 pt: chain rule).
  $h'(7) = f'(g(7)) \cdot g'(7) = f'(0) \cdot 8 = \frac{3}{2} \cdot 8 = 12$ (1 pt: correct answer 12, using the table values g(7)=0, f'(0)=3/2, g'(7)=8).

Solution 5

(b)

Mark scheme

- $k''(x) = 2f(x)f'(x)g(x) + (f(x))^2 g'(x)$ (1 pt: product/chain rule applied to k'(x)=(f(x))^2 g(x)). $k''(4) = 2f(4)f'(4)g(4) + (f(4))^2 g'(4) = 2(4)(3)(-3) + (4)^2(2) = -72 + 32 = -40$ (1 pt: k''(4)=-40 with supporting work using the table values f(4)=4, f'(4)=3, g(4)=-3, g'(4)=2). Since k''(4)=-40<0 and k'' is continuous, the graph of k is concave down at x=4 (1 pt: answer with reason, consistent with the declared nonzero value of k''(4)).

Solution 5

(c)

Mark scheme

- By the Fundamental Theorem of Calculus, $m(2) = 5(2)^3 + \int_0^2 f'(t)\,dt = 40 + (f(2) - f(0)) = 40 + (7 - 10) = 37$ (1 pt: answer 37 with supporting work equivalent to $5 \cdot 8 + (f(2) - f(0))$).

Solution 5

(d)

Mark scheme

- $m'(x) = 15x^2 + f'(x)$ (1 pt: considers m'(x)).
  $m'(2) = 15(2)^2 + f'(2) = 60 + (-8) = 52$ (1 pt: m'(2) with supporting work, using f'(2)=-8 from the table; an unsupported m'(2)=52 does not earn this point). Since $m'(2) = 52 > 0$, the graph of m is increasing at x=2 (1 pt: answer with justification, consistent with the declared value of m'(2)).

Question 1

2022 ▪ Q1

From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

(a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

(d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by

Question 1

$$N(t) = \int_a^t (A(x) - 400)\, dx,$$

where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Solution 1

(a)

Mark scheme

- The total number of vehicles arriving from 6 A.M. to 10 A.M. is given by the definite integral $\int_1^5 A(t)\,dt$. Full credit requires a definite integral with the correct lower limit $t = 1$ and upper limit $t = 5$; the integrand should be $A(t)$ (equivalently $|A(t)|$, since $A(t) \geq 0$ on this interval). Do not evaluate the integral.

Solution 1

(b) Mark scheme

- Average value $= \dfrac{1}{5-1}\int_1^5 A(t)\,dt = \dfrac{1}{4}(1502.147865) = 375.536966.$ The average rate at which vehicles arrive at the toll plaza from 6 A.M. to 10 A.M. is about 375.537 (or 375.536) vehicles per hour. One point for using the correct average value formula $\frac{1}{b-a}\int_a^b A(t)\,dt$ with $a = 1$, $b = 5$; one point for the correct answer, accurate to three decimal places.

Solution 1

(c)

Mark scheme

- $A'(1) = 148.947272 > 0$, so the rate at which vehicles arrive at the toll plaza is increasing at $t = 1$ (6 A.M.). One point for considering $A'(1)$; one point for the correct answer (increasing) together with the reason $A'(1) > 0$.

Solution 1

(d)

Mark scheme

- $N'(t) = A(t) - 400 = 0 \Rightarrow A(t) = 400 \Rightarrow t = a = 1.469372$ or $t = b = 3.597713$. Evaluating $N(t) = \int_a^t (A(x) - 400)\,dx$ at the critical points and the right endpoint: $N(a) = 0$, $N(b) = 71.254129$, $N(4) = 62.338346$. The greatest number of vehicles in line is 71 (attained at $t = b$). Points: (1) considers $N'(t) = 0$; (2) correctly finds both $t = a$ and $t = b$; (3) correct answer of 71 (or 71.254...); (4) justification via a candidates-test table comparing $N(a)$, $N(b)$, and $N(4)$.

Question 2

2022 ▪ Q2

[Graph: a coordinate plane with x-axis from -2 to 1 and y-axis from -1 to 1 (tick marks labeled -1 on y-axis, 1 on y-axis; -2, -1, O, 1 on x-axis). A curve labeled implicitly by the two functions below starts at $(-2, 0)$, dips down to a minimum below the x-axis between $x = -1$ and $x = 0$, rises back up crossing near the origin, and continues upward, ending with a marked point just above and to the right of $(1, 1)$.]

Let f and g be the functions defined by $f(x) = \ln(x + 3)$ and $g(x) = x^4 + 2x^3$. The graphs of f and g, shown in the figure above, intersect at $x = -2$ and $x = B$, where $B > 0$.

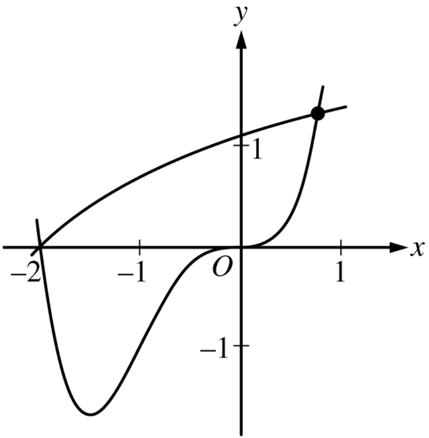
(a) Find the area of the region enclosed by the graphs of f and g.

(b) For $-2 \leq x \leq B$, let $h(x)$ be the vertical distance between the graphs of f and g. Is h increasing or decreasing at $x = -0.5$? Give a reason for your answer.

Question 2

(c) The region enclosed by the graphs of f and g is the base of a solid. Cross sections of the solid taken perpendicular to the x-axis are squares. Find the volume of the solid.
(d) A vertical line in the xy-plane travels from left to right along the base of the solid described in part (c). The vertical line is moving at a constant rate of 7 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$.

Question 2



Solution 2

(a)

Mark scheme

- $\ln(x+3) = x^4 + 2x^3 \Rightarrow x = -2,\ x = B = 0.781975$. Area $= \int_{-2}^{B} (f(x) - g(x))\,dx = \int_{-2}^{B} \left(\ln(x+3) - (x^4 + 2x^3)\right) dx = 3.603548$. The area of the region is 3.604 (or 3.603). Points: (1) correct integrand $f(x) - g(x)$ (or an equivalent form such as $|f(x) - g(x)|$ or $g(x) - f(x)$); (2) correct limits of integration, -2 to B; (3) correct answer.

Solution 2

(b)

Mark scheme

- $h(x) = f(x) - g(x)$, so $h'(x) = f'(x) - g'(x)$.
  $h'(-0.5) = f'(-0.5) - g'(-0.5) = -0.6$ (or -0.599). Since $h'(-0.5) < 0$, h is decreasing at $x = -0.5$. Points: (1) considers $h'(-0.5)$ (or equivalently $f'(-0.5) - g'(-0.5)$); (2) correct answer (decreasing) with reason.

Solution 2

(c)

Mark scheme

- Volume $= \int_{-2}^{B} (f(x) - g(x))^2 \, dx = 5.340102$. The volume of the solid is 5.340. Points: (1) correct integrand $k(f(x) - g(x))^2$ with $k = 1$ in a definite integral; (2) correct answer.

Solution 2

(d) Mark scheme

- The cross section has area $A(x) = (f(x) - g(x))^2$. By the chain rule, $\dfrac{d}{dt}[A(x)] = \dfrac{dA}{dx} \cdot \dfrac{dx}{dt}$. At $x = -0.5$: $\dfrac{d}{dt}[A(x)]\Big|_{x=-0.5} = A'(-0.5) \cdot 7 = -9.271842$. At $x = -0.5$, the area of the cross section is changing at a rate of about -9.272 (or -9.271) square units per second. Points: (1) correct chain-rule expression $\frac{dA}{dx} \cdot \frac{dx}{dt}$ (equivalently $A'(x) \cdot dx/dt$); (2) correct answer.

Question 3

2022 ▪ Q3

[Graph titled "Graph of f'": a coordinate plane with x-axis from 0 to 7 (integer gridlines at $1, 2, 3, 4, 5, 6, 7$) and y-axis from -2 to 2 (gridlines at $-2, -1, 1, 2$). The graph consists of a semicircle below the x-axis from $x = 0$ to $x = 4$, dipping down to a minimum of about -2 near $x = 2$, returning to 0 at $x = 4$. From $x = 4$ to $x = 6$ a line segment rises from $(4, 0)$ to the marked point $(6, 2)$. From $x = 6$ to $x = 7$ a line segment falls from $(6, 2)$ to the marked point $(7, 1)$.]

Let f be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

(a) Find $f(0)$ and $f(5)$.

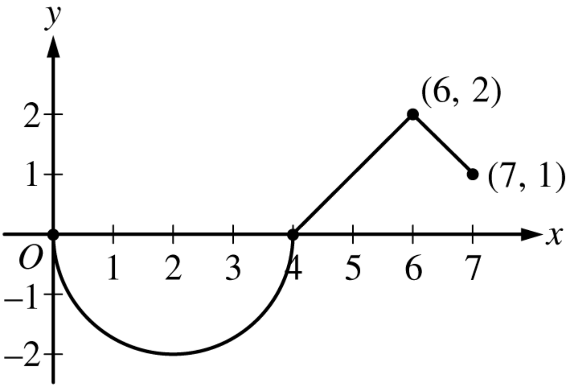
(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

Question 3

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

Question 3



Graph of f'

Solution 3

(a)

Mark scheme

- $f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 + 2\pi$ (the semicircle over $[0,4]$ has radius 2 and lies below the axis, so $\int_0^4 f'(x)\,dx = -2\pi$). $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$. Points: (1) area of either region ($\int_0^4 f'$ or $\int_4^5 f'$); (2) correct $f(0) = 3 + 2\pi$; (3) correct $f(5) = 7/2$.

Solution 3

(b)

Mark scheme

- The graph of f has points of inflection at $x = 2$ and $x = 6$, because f' changes from decreasing to increasing at $x = 2$ (the bottom of the semicircle) and from increasing to decreasing at $x = 6$ (the corner/peak of the graph of f' at $(6, 2)$). Points: (1) states both $x = 2$ and $x = 6$; (2) correct justification tied to the graph of f' (e.g., discussing where the slope/behavior of f' changes, or citing $x = 2, 6$ as locations of local extrema on the graph of f').

Solution 3

(c)

Mark scheme

- $g(x) = f(x) - x \Rightarrow g'(x) = f'(x) - 1$. Since $g'(x) \leq 0 \iff f'(x) \leq 1$, and from the graph $f'(x) \leq 1$ for $0 \leq x \leq 5$, the graph of g is decreasing on the interval $0 \leq x \leq 5$ (because $g'(x) \leq 0$ there). Points: (1) correct $g'(x) = f'(x) - 1$ (or equivalent); (2) correct interval $[0, 5]$ with reason.

Solution 3

(d) Mark scheme

- g is continuous, $g'(x) < 0$ for $0 < x < 5$, and $g'(x) > 0$ for $5 < x < 7$, so the absolute minimum occurs at $x = 5$: $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$. The absolute minimum value of g on $[0, 7]$ is $-\frac{3}{2}$. (Alternative: a Candidates Test comparing $g(0) = 3 + 2\pi$, $g(5) = -3/2$, $g(7) = -1/2$ gives the same result.) Points: (1) considers $g'(x) = 0$, isolating $x = 5$ as the only critical number on $(0, 7)$; (2) correct answer $-3/2$ with justification.

Question 3

2018 ▪ Q3

[Graph of g: a piecewise-defined curve on axes with x from -5 to 6 (gridlines every 1 unit) and y from -3 to 8 (gridlines every 1 unit). The curve is constant at $y = -3$ for $-5 \leq x \leq -2$; rises linearly from $(-2, -3)$ to $(-1, 0)$; is constant at $y = 0$ for $-1 \leq x \leq 0$; rises linearly from $(0, 0)$ to $(1, 2)$; is constant at $y = 2$ for $1 \leq x \leq 3$; then follows a smooth curve for $3 \leq x \leq 6$ that dips down to a minimum value of 0 at $x = 4$ and rises back up, passing through approximately $(5, 1)$ and reaching $(6, 8)$.]

Graph of g

The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \leq x < 3$, and $g(x) = 2(x - 4)^2$ for $3 \leq x \leq 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

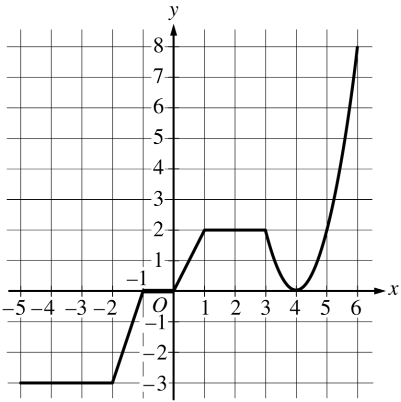
Question 3

(b) Evaluate $\int_1^6 g(x)\, dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

Question 3



Graph of g

Solution 3

(a)

Mark scheme

- $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^{1} g(x)\,dx = 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$. (1 pt: the integral expression for $f(-5)$; 1 pt: the answer $\frac{25}{2}$.)

Solution 3

(b)

Mark scheme

- $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx = \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx = 4 + \left[\frac{2}{3}(x-4)^3\right]_3^6 = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$. (1 pt: splits the integral at $x = 3$; 1 pt: antiderivative $\frac{2}{3}(x-4)^3$ of $2(x-4)^2$; 1 pt: the answer 10.)

Solution 3

(c)

Mark scheme

- The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$, because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing (so $f''(x) = g'(x) > 0$) on those intervals. (1 pt: the correct intervals; 1 pt: the reason.)

Solution 3

(d)

Mark scheme

- The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$ (so f'' changes sign there). (1 pt: the answer $x = 4$; 1 pt: the reason.)

Question 2

2017 ▪ Q2

When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \leq 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \leq 12,$$

Question 2

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

(a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

(b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

(c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

(d) How many pounds of bananas are on the display table at time $t = 8$?

Question 3

2017 ▪ Q3

[Graph of f', labeled "Graph of f'"; consists of a semicircle and three line segments. x-axis x, y-axis y. A line segment starts at the labeled point $(-6, 2)$ and decreases linearly to the point $(-2, 0)$ on the x-axis. From $(-2, 0)$ to $(0, 0)$ the graph is a semicircle dipping below the x-axis (minimum around $(-1, -1)$) and returning to the x-axis at $(0, 0)$. From $(0, 0)$ the graph rises linearly to the labeled point $(3, 2)$. From $(3, 2)$ the graph decreases linearly to the point $(5, 0)$ on the x-axis. Gridlines/tick marks are shown at integer x-values and at $y = 1$.]

The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

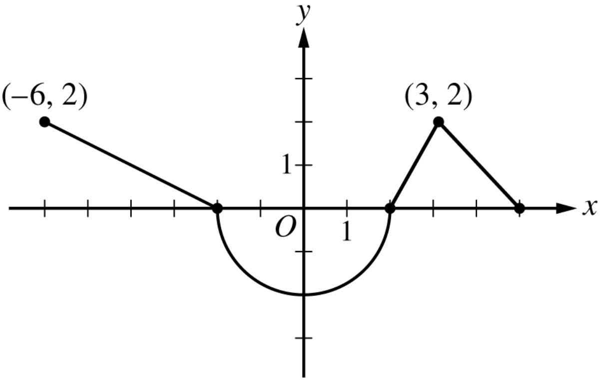
(b) On what intervals is f increasing? Justify your answer.

Question 3

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

Question 3



Graph of f'

Question 5

2015 ▪ Q5

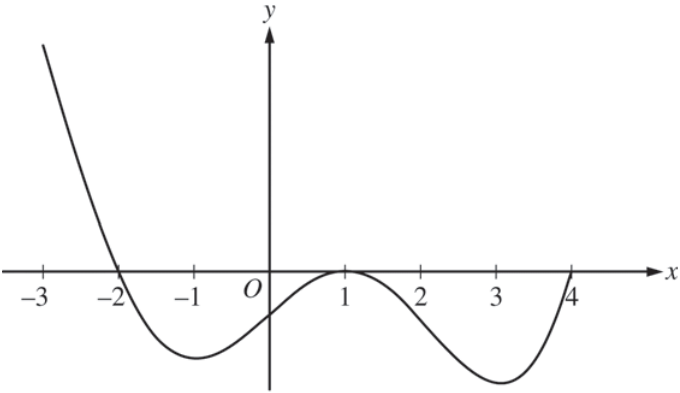
[Graph of f': axes labeled y (vertical) and x (horizontal), with tick marks on the x-axis at $-3, -2, -1, O, 1, 2, 3, 4$. The curve begins high on the left (above $x = -3$), decreases steeply, crosses the x-axis just before $x = -1$, continues down to a local minimum between $x = -1$ and O, rises back up crossing the x-axis near $x = 1$, reaches a local maximum between $x = 1$ and $x = 2$, then decreases again crossing the x-axis before $x = 3$, reaches a local minimum between $x = 3$ and $x = 4$, then rises steeply up to $x = 4$. Labeled "Graph of f'".]

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

Question 5

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.
(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.
(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.
(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

Question 5



Graph of f'

Solution 5

(a)

Mark scheme

- $f'(x) = 0$ at $x = -2$, $x = 1$, and $x = 4$. $f'(x)$ changes from positive to negative only at $x = -2$. Therefore f has a relative maximum at $x = -2$ (this is the only x-coordinate at which f has a relative maximum). Points: 1 for identifying $x = -2$, 1 for the answer with correct reason (sign change of f' from + to −).

Solution 5

(b)

Mark scheme

- The graph of f is concave down and decreasing on the intervals $-2 < x < -1$ and $1 < x < 3$, because f' is decreasing and negative on these intervals. Points: 1 for identifying the correct intervals $(-2, -1)$ and $(1, 3)$, 1 for the correct reason (f' decreasing gives concave down; f' negative gives decreasing).

Solution 5

(c)

Mark scheme

- The graph of f has points of inflection at $x = -1$ and $x = 3$, because f' changes from decreasing to increasing at these points. The graph of f also has a point of inflection at $x = 1$, because f' changes from increasing to decreasing at this point. So the inflection x-coordinates are -1, 1, and 3. Points: 1 for identifying $x = -1$, 1, and 3, 1 for the correct reason (f' changes from increasing to decreasing, or decreasing to increasing, at each point).

Solution 5

(d)

Mark scheme

- f(x) = 3 + $\int$ ^x f'(t) dt (using f(1) = 3). f(4) = 3 + $\int$ ^4 f'(t) dt = 3 + (−12) = −9 (the integral is the negative of the area 12 on [1, 4], since f' is negative there). f(−2) = 3 + $\int$ ^(−2) f'(t) dt = 3 − $\int$ _(−2)^1 f'(t) dt = 3 − (−9) = 12 (the integral over [−2, 1] is the negative of the area 9, since f' is negative there). Points: 1 for the correct integrand f'(t) in an expression for f, 1 for a correct expression for f(x) (f(x) = 3 + $\int$ ^x f'(t) dt), 1 for both correct final values f(4) = −9 and f(−2) = 12.