Past-paper walk-through

Extreme Value Theorem, Global Versus Local Extrema, and Critical Points ▪ 5.2

eXercise

Questions in this set

- 2024 Q3
- 2024 Q4
- 2021 Q3
- 2021 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

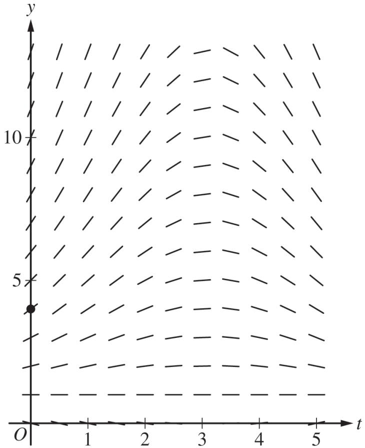
2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H - 1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t = 0)$. It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.

[Slope field graph; t-axis (horizontal) labeled 1 to 5, y-axis (vertical) labeled 5 and 10; a dot is plotted at $(0, 4)$; short line segments show the slope at each grid point — segments are steep and negatively-sloped (like "") in the upper-right region, nearly flat in the lower region, and forward-sloped (like "/") in the upper-left region, consistent with the given differential equation.]

Question 3



Question 3

2024 ▪ Q3

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(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t=0)$. It is known that $H(0) = 4$.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$ with initial condition $H(0) = 4$.

Solution 3

(a) Mark scheme

- The solution curve must pass through $(0, 4)$ and follow the given slope field: it rises (following the steep positive-then-shallowing slopes), reaches a relative maximum near $t = \pi \approx 3.14$ (height about 9.4, matching part (b)'s critical point), then decreases for $\pi < t < 5$, staying consistent with the marked segments and extending to at least $t = 4.5$ with no obvious conflicts with the drawn slope lines. 1 point awarded only if the curve passes through $(0, 4)$, extends to at least $t = 4.5$, and has no obvious conflicts with the given slope lines (only the portion inside the given field is considered).

Solution 3

(b) Mark scheme

- Since $H(t) > 1$ on $0 < t < 5$, $\dfrac{dH}{dt} = 0$ requires $\cos\left(\dfrac{t}{2}\right) = 0$, which gives $t = \pi$. For $0 < t < \pi$, $\dfrac{dH}{dt} > 0$, and for $\pi < t < 5$, $\dfrac{dH}{dt} < 0$; therefore $t = \pi$ is the location of a relative maximum of H (the depth of seawater). 1 point for considering the sign of dH/dt (or equivalently $\cos(t/2)$), 1 point for identifying $t = \pi$ as the critical point, 1 point (only earnable with the first) for the correct classification 'relative maximum' with justification via a sign analysis of dH/dt on either side of $t = \pi$ (or an equivalent Second Derivative Test).

Solution 3

(c) Mark scheme

- Separate variables: $\dfrac{dH}{H-1} = \dfrac{1}{2}\cos\left(\dfrac{t}{2}\right)\,dt$. Integrate both sides: $\ln|H-1| = \sin\left(\dfrac{t}{2}\right) + C$. Apply $H(0)=4$: $\ln|4-1| = \sin(0) + C \Rightarrow C = \ln 3$; since $H(0) = 4 > 1$, $|H-1| = H-1$, so $\ln(H-1) = \sin(t/2) + \ln 3$. Solve for H: $H - 1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)}$, so $H(t) = 1 + 3e^{\sin(t/2)}$. 1 point for correct separation of variables, 1 point for one correct antiderivative (either side), 1 point for the second correct antiderivative, 1 point for including the constant of integration and correctly using the initial condition $H(0) = 4$ (eligible only if the separation point and at least one antiderivative point were earned), 1 point for the

Solution 3

fully solved final answer $H(t) = 1 + 3e^{\sin(t/2)}$ (eligible only if the first four points were earned; a response with no separation of variables earns 0 of 5).

Question 4

2024 ▪ Q4

[Graph of f; x-axis labeled -6 to 7, y-axis labeled -2 to 3; the labeled point $(-6, 0.5)$ is the left endpoint of the curve; the curve rises to a horizontal tangent (peak) at $x = -2$ reaching just above $y = 2$, then descends, crossing the x-axis near $x = 4$, and continues as a straight line down to the labeled point $(6, -1)$ and beyond; the shaded region R lies in the second quadrant between the curve, the vertical line $x = -6$, and the x- and y-axes, spanning roughly $-6 \le x \le 0$.]

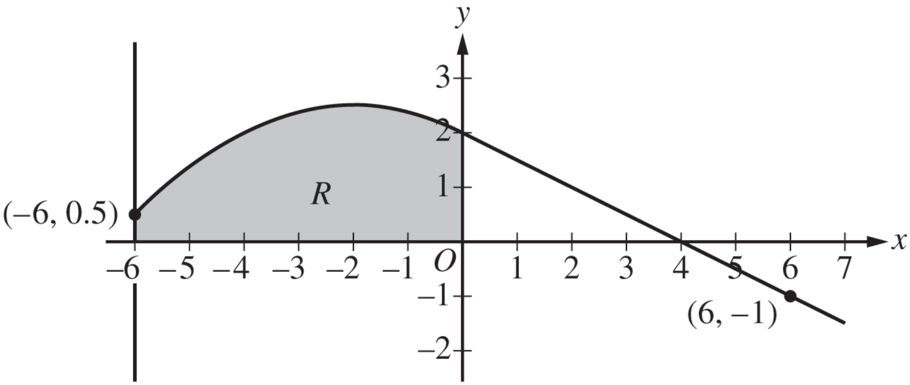
The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

(a) The function g is defined by $g(x) = \int_0^x f(t)\, dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

Question 4

(b) For the function g defined in part (a), find all values of x in the interval $0 \leq x \leq 6$ at which the graph of g has a critical point. Give a reason for your answer.
(c) The function h is defined by $h(x) = \int_{-6}^{x} f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Question 4



Graph of f

Solution 4

(a) Mark scheme

- Using the graph of f and $g(x) = \int_0^x f(t)\,dt$:
  $g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^{0} f(t)\,dt = -12$ (since region R, the area under f from -6 to 0, has area 12, given in the problem). $g(4) = \int_0^4 f(t)\,dt = \frac{1}{2}(4)(2) = 4$ (triangle area under the linear piece of f from $x = 0$, where $f(0) = 2$, down to where it crosses the x-axis at $x = 4$).
  $g(6) = \int_0^6 f(t)\,dt = \frac{1}{2}(4)(2) - \frac{1}{2}(2)(1) = 4 - 1 = 3$ (adding the negative triangle from $x = 4$ to $x = 6$, where $f(6) = -1$). 1 point each for $g(-6) = -12$, $g(4) = 4$, $g(6) = 3$ (supporting work not required, but if shown it must be correct to earn the point; unlabeled values are read in the order $g(-6), g(4), g(6)$).

Solution 4

Solution 4

(b)

Mark scheme

- By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Setting $g'(x) = f(x) = 0$ on $0 \leq x \leq 6$ gives $x = 4$ (where the graph of f crosses the x-axis, matching part (a)). Therefore the graph of g has a critical point at $x = 4$. 1 point for explicitly making the connection $g' = f$, 1 point for the answer $x = 4$ with a valid reason (e.g. $f(4) = 0$ and f changes sign there).

Solution 4

(c) Mark scheme

- $h(x) = \int_{-6}^{x} f'(t)\,dt$, so by the Fundamental Theorem of Calculus $h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5 = -1.5$. $h'(x) = f'(x)$, so $h'(6) = f'(6) = -\dfrac{1}{2}$ (the slope of the linear piece of f on $0 \le x \le 7$, which passes through $(0, 2)$ and $(6, -1)$). $h''(x) = f''(x)$, so $h''(6) = f''(6) = 0$ (since f is linear on $[0, 7]$, its second derivative is 0 there). 1 point for using the Fundamental Theorem of Calculus to evaluate $h(6)$ as $f(6) - f(-6)$, 1 point for the value $h(6) = -1.5$ with supporting work, 1 point for $h'(6) = -1/2$, 1 point for $h''(6) = 0$.

Question 3

2021 ▪ Q3

[Graph showing a curve in the first quadrant starting at the origin O, rising to a rounded peak, then curving back down to meet the x-axis; x-axis and y-axis labeled, no numeric scale marks shown.]

A company designs spinning toys using the family of functions $y = cx\sqrt{4-x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

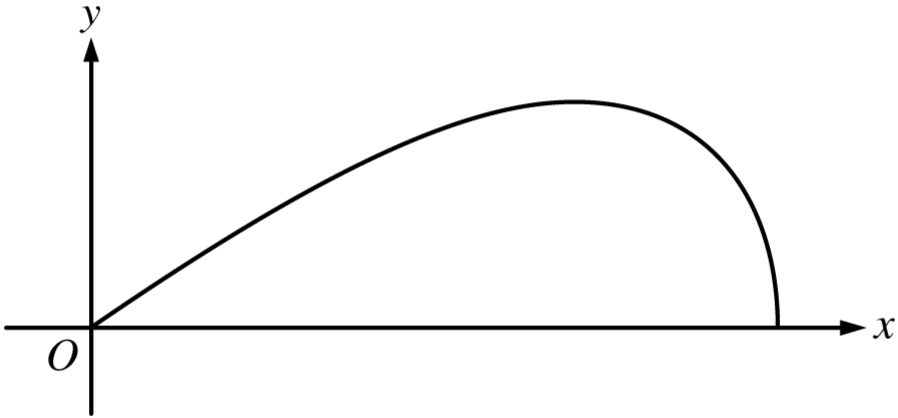
(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$ for $c = 6$.

Question 3

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c(4 - 2x^2)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Question 3



Solution 3

(a) Mark scheme

- For $c = 6$: $6x\sqrt{4-x^2} = 0 \Rightarrow x = 0, 2$, so Area $= \displaystyle\int_0^2 6x\sqrt{4-x^2}\,dx$. Using $u = 4 - x^2$: $\int_0^2 6x\sqrt{4-x^2}\,dx = 3\int_0^4 u^{1/2}\,du = 2u^{3/2}\Big|_0^4 = 2\cdot 8 = 16$. The area of the region is 16 square inches. Point 1: presenting $cx\sqrt{4-x^2}$ or $6x\sqrt{4-x^2}$ as the integrand of a definite integral (limits need not be correct). Point 2: a correct antiderivative of the form $Ax\sqrt{4-x^2}$… [presented as $Au^{3/2}$-type expression] for any nonzero constant (independent of point 1). Point 3: the final answer 16 (or equivalent), only earned if point 2 was earned, using the correct limits.

Solution 3

(b) Mark scheme

- The largest cross-sectional radius occurs where $\dfrac{dy}{dx} = 0$:
  $\dfrac{dy}{dx} = \dfrac{c(4-2x^2)}{\sqrt{4-x^2}} = 0 \Rightarrow x = \sqrt{2}$. Then $y = c\sqrt{2}\sqrt{4-(\sqrt{2})^2} = 2c$. Setting $2c = 1.2$ gives $c = 0.6$. Point 1: setting $\frac{dy}{dx} = 0$ (or the equivalent $c(4-2x^2) = 0$); an unsupported $x = \sqrt{2}$ alone does not earn it. Point 2: the final answer $c = 0.6$ with supporting work (can be earned without point 1).

Solution 3

(c) Mark scheme

- Volume $= \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2 \int_0^2 x^2(4-x^2)\,dx =$ $\pi c^2 \int_0^2 (4x^2 - x^4)\,dx = \pi c^2 \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \pi c^2\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi c^2}{15}$. Setting $\frac{64\pi c^2}{15} = 2\pi$ gives $c^2 = \frac{15}{32}$, so $c = \sqrt{15/32}$. Point 1: integrand of the form $A(x\sqrt{4-x^2})^2$ in a definite integral, any limits, any nonzero constant A (mishandling c makes it ineligible for point 4). Point 2: correct limits $x = 0, 2$ and constant π (not 2π) as part of an integral — earnable without point 1. Point 3: correct antiderivative of the presented integrand, form $A(x\sqrt{4-x^2})^2$-derived

Solution 3

polynomial, for any nonzero A — needed for point 4. Point 4: final answer $c = \sqrt{15/32}$ (need not be simplified); cannot be earned without point 3.

Question 5

2021 ▪ Q5

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y \cos x}{4y - \sin x}$.

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

(c) For $0 \leq x \leq \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

Solution 5

(a) Mark scheme

- Implicitly differentiate $2y^2 - 6 = y \sin x$: $\frac{d}{dx}(2y^2 - 6) = \frac{d}{dx}(y \sin x) \Rightarrow 4y\frac{dy}{dx} = \frac{dy}{dx}\sin x + y \cos x$. Then $4y\frac{dy}{dx} - \frac{dy}{dx}\sin x = y \cos x \Rightarrow \frac{dy}{dx}(4y - \sin x) = y \cos x \Rightarrow \frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$, as required. Point 1: correct implicit differentiation of $2y^2 - 6 = y \sin x$ (alternative notations for dy/dx such as y' are fine) — required for point 2. Point 2: verification, i.e. reaching $\frac{dy}{dx}(4y - \sin x) = y \cos x$ is sufficient with no subsequent errors.

Solution 5

(b)

Mark scheme

- At the point $(0, \sqrt{3})$: $\dfrac{dy}{dx} = \dfrac{\sqrt{3}\cos 0}{4\sqrt{3} - \sin 0} = \dfrac{1}{4}$. An equation for the tangent line is $y = \sqrt{3} + \dfrac{1}{4}x$. Point 1: any correct tangent line equation; no supporting work is required and the slope value need not be simplified.

Solution 5

(c) Mark scheme

- $\dfrac{dy}{dx} = \dfrac{y\cos x}{4y - \sin x} = 0 \Rightarrow y\cos x = 0$ and $4y - \sin x \neq 0$. Since $y\cos x = 0$ and $y > 0 \Rightarrow \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}$ (on $0 \leq x \leq \pi$). When $x = \pi/2$: $y\sin(\pi/2) = 2y^2 - 6 \Rightarrow y = 2y^2 - 6 \Rightarrow 2y^2 - y - 6 = 0 \Rightarrow (2y+3)(y-2) = 0 \Rightarrow y = 2$ (since $y > 0$). Check: $4y - \sin x = 8 - 1 = 7 \neq 0$. The tangent line is horizontal at $\left(\dfrac{\pi}{2}, 2\right)$. Point 1: sets $\frac{dy}{dx} = 0$ (or equivalently $y\cos x = 0$ or $\cos x = 0$). Point 2: commits to $x = \pi/2$ only (any extraneous additional x-values forfeit this point; y-values presented are not considered here). Point 3: $y = 2$ (not earned without point 2; coordinates need not be presented as an ordered pair).

Solution 5

(d) Mark scheme

- $\dfrac{d^2y}{dx^2} = \dfrac{(4y - \sin x)\left(\frac{dy}{dx}\cos x - y\sin x\right) - (y\cos x)\left(4\frac{dy}{dx} - \cos x\right)}{(4y - \sin x)^2}$. At $x = \pi/2, y = 2$ (with $dy/dx = 0$): $\dfrac{d^2y}{dx^2} = \dfrac{(7)(-2) - (0)(0)}{49} = \dfrac{-2}{7} < 0$. Since $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$ at $\left(\frac{\pi}{2}, 2\right)$, f has a RELATIVE MAXIMUM there (Second Derivative Test). [Alternate solution via First Derivative Test: near $(\pi/2, 2)$, $4y - \sin x > 0$ and $y > 0$, so $\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$ changes from positive to negative at $x = \pi/2$, giving a relative maximum by the First Derivative Test.] Point 1: attempt to use the quotient (or product) rule to find d^2y/dx^2 [or: considers the

Solution 5

sign of $4y - \sin x$]. Point 2: correctly finds d^2y/dx^2 and evaluates it as negative at the point [or: shows dy/dx changes sign $+$ to $-$ at $x = \pi/2$ — can be earned without point 1]. Point 3: correct conclusion (relative maximum) with justification, consistent with a reported nonzero value/sign obtained using $dy/dx = 0$; a response concluding "minimum" earns no point-3 credit.