Past-paper walk-through

Using L'Hospital's Rule for Determining Limits of Indeterminate Forms ▪ 4.7

eXercise

Questions in this set

- 2023 Q4
- 2021 Q4
- 2019 Q3
- 2019 Q6
- 2018 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 4

2023 ▪ Q4

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.
[Graph of f': x-axis from -2 to 8 with gridlines at every integer, y-axis from -2 to 2 with gridlines at every integer. The graph consists of: a line segment from $(-2, 2)$ down to $(0, -2)$; a line segment from $(0, -2)$ up to $(4, 2)$; and a semicircle (bulging downward, below the x-axis) connecting $(4, 2)$ down through the x-axis and back up to $(8, 2)$, dipping to a minimum of about 0 at $x = 6$. Solid dots are marked at $(-2, 2)$, $(0, -2)$, $(4, 2)$, and $(8, 2)$.]
(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

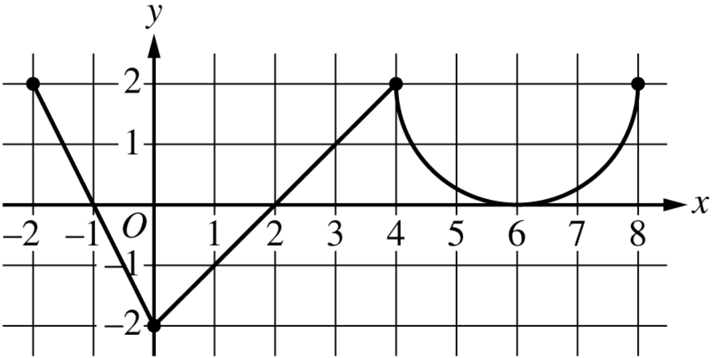
Question 4

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

(c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Question 4



Graph of f'

Solution 4

(a)

Mark scheme

- $f'(x) > 0$ on (2,6) and $f'(x) > 0$ on (6,8), so f'(x) does not change sign at x=6; therefore f has neither a relative maximum nor a relative minimum at x=6 (1 pt: correct answer — 'neither' — with the reason that f'(x) does not change sign at x=6).

Solution 4

(b)

Mark scheme

- The graph of f is concave down on (-2,0) and (4,6) because f' is decreasing on these intervals (1 pt: correct intervals (-2,0) and (4,6) — with or without endpoints; 1 pt: reason, must correctly discuss the behavior/slopes of f').

Solution 4

(c) Mark scheme

- Because f is differentiable, hence continuous, at x=2, $\lim_{x\to 2} f(x) = f(2) = 1$; so $\lim_{x\to 2}(6f(x) - 3x) = 6(1) - 3(2) = 0$ and $\lim_{x\to 2}(x^2 - 5x + 6) = 0$ (1 pt: both limits of numerator and denominator shown to equal 0; a limit presented as literally equal to 0/0 does not earn this point). Since the limit is of indeterminate form 0/0, L'Hospital's Rule applies (1 pt: uses L'Hospital's Rule — at least one correct derivative shown in the limit of a ratio of derivatives). $\lim_{x\to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x\to 2} \dfrac{6f'(x) - 3}{2x - 5} = \dfrac{6(0) - 3}{2(2) - 5} = \dfrac{-3}{-1} = 3$ (1 pt: correct answer 3 with supporting work).

Solution 4

(d)

Mark scheme

- $f'(x) = 0 \Rightarrow x = -1, 2, 6$ (1 pt: considers f'(x)=0 — merely listing the zeros of f' without stating f'=0 is not sufficient). Since f is continuous on [-2,8], the candidates for the location of an absolute minimum are x=-2,-1,2,6,8, with $f(-2) = 3$, $f(-1) = 4$, $f(2) = 1$, $f(6) = 7 - \pi$, $f(8) = 11 - 2\pi$ (1 pt: justification — evaluate f at all critical points and both endpoints, or validly eliminate some by argument). The absolute minimum value of f is $f(2) = 1$ (1 pt: answer — must state the minimum VALUE 1, not merely its location x=2).

Question 4

2021 ▪ Q4

[Graph of f on a grid, x-axis from -4 to 6, y-axis from -4 to 6, gridlines every 2 units on the y-axis and every 2 units on the x-axis; the graph consists of four connected line segments: from $(-4, 0)$ up to $(-2, 6)$, then down to $(0, 4)$, then down to $(2, -4)$, then up to $(6, 0)$. Labeled "Graph of f".]

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

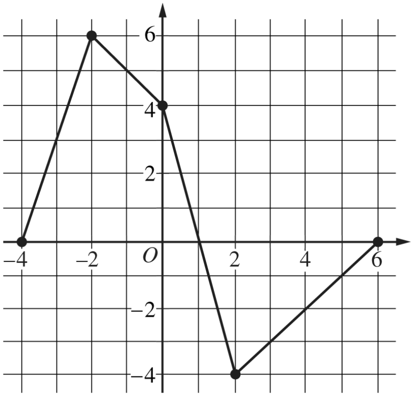
(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

Question 4

(c) Find $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- $G'(x) = f(x)$ (the "global point," worth 1 point, earnable anywhere in parts (a)-(c) by writing $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$, $G'(3) = f(3)$, or $G'(2) = f(2)$). The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals (intervals may include one or both endpoints). Point (part a, 1 point): correct intervals with the reason that $G' = f$ is increasing there. Point (global, 1 point, folded into this leaf): correctly stating $G'(x) = f(x)$ anywhere in the response.

Solution 4

(b)

Mark scheme

- $P(x) = G(x)f(x) \Rightarrow P'(x) = G(x)f'(x) + f(x)G'(x)$. Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ (and $f'(3) = 1$, the slope of the line segment from $(2, -4)$ to $(6, 0)$): $P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$. Point 1: correct application of the product rule in terms of x or evaluated at 3 (once earned, cannot be lost). Point 2: correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$. Point 3: the final answer 5.5 (needs points 1 and 2 first; simplification not required).

Solution 4

(c) Mark scheme

- $\lim_{x\to 2}(x^2 - 2x) = 0$ and, since G is continuous, $\lim_{x\to 2} G(x) = \int_0^2 f(t)\,dt = 0$, so the limit is the indeterminate form $0/0$. By L'Hospital's Rule, $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x} = \lim_{x\to 2} \dfrac{G'(x)}{2x - 2} = \lim_{x\to 2} \dfrac{f(x)}{2x - 2} = \dfrac{f(2)}{2} = \dfrac{-4}{2} = -2$. Point 1: shows $\lim(x^2 - 2x) = 0$ and $\lim G(x) = 0$, and shows the ratio of the two derivatives $G'(x)$ and $2x - 2$ (may be shown as an evaluation at $x = 2$). Point 2: evaluates the limit correctly with appropriate limit notation, including $\lim_{x\to 2} \frac{G'(x)}{2x-2}$ or $\lim_{x\to 2} \frac{f(x)}{2x-2}$; any linkage error (e.g. writing $\frac{G'(x)}{2x-2} = \frac{f(2)}{2}$ without the limit) forfeits this point.

Solution 4

(d) Mark scheme

- $G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$. Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$. Yes — since $G'(x) = f(x)$, G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$, so the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \dfrac{8}{3}$. Point 1: presents at least a difference and a quotient with correct evaluation (numerical simplification not required, but if simplified must be correct). Point 2: correct conclusion (earnable even with an incorrect/no evaluation of the average rate of change, as long as MVT's conditions and conclusion are stated — an explicit MVT citation is not required).

Solution 4

Question 3

2019 ▪ Q3

The continuous function *f* is defined on the closed interval $-6 \leq x \leq 5$. The figure below shows a portion of the graph of *f*, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3,\ 3 - \sqrt{5})$ is on the graph of *f*.
[Graph of *f*; *x*-axis from -2 to 5 (gridlines shown from -2 to 5), *y*-axis from -1 to 3. A line segment goes from a closed point at $(-2, 1)$ down to a closed point at the origin $(0, 0)$. From $(0, 0)$ a line segment rises to a closed point at $(2, 3)$. From $(2, 3)$ a curve (quarter circle centered at $(5, 3)$) decreases and flattens out, passing through $(3, 3 - \sqrt{5})$, ending at a closed point at $(5, 0)$ on the *x*-axis. Labeled "Graph of *f*".]

(a) If $\displaystyle\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\displaystyle\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

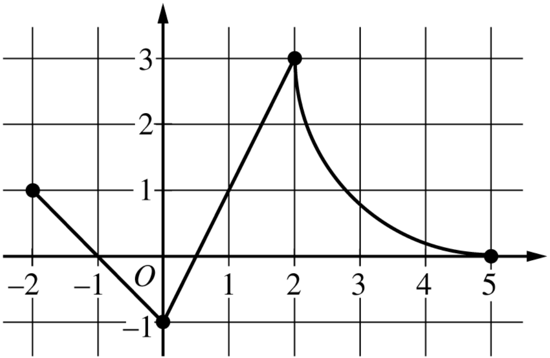
Question 3

(b) Evaluate $\int_3^5 \left(2f'(x) + 4\right) dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.

Question 3



Graph of f

Solution 3

(a) Mark scheme

- $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$. Given $\int_{-6}^{5} f(x)\,dx = 7$ and (from the given graph of f) $\int_{-2}^{5} f(x)\,dx = 2 + \left(9 - \frac{9\pi}{4}\right) = 11 - \frac{9\pi}{4}$, so $7 = \int_{-6}^{-2} f(x)\,dx + \left(11 - \frac{9\pi}{4}\right)$, giving $\int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$. Points: 1 for splitting the integral correctly ($\int_{-6}^{5} = \int_{-6}^{-2} + \int_{-2}^{5}$), 1 for the correct value of $\int_{-2}^{5} f(x)\,dx$, 1 for the final answer $\frac{9\pi}{4} - 4$.

Solution 3

(b)

Mark scheme

- $\int_3^5 (2f'(x) + 4)\, dx = 2\int_3^5 f'(x)\, dx + \int_3^5 4\, dx = 2(f(5) - f(3)) + 4(5-3) = 2\left(0 - (3 - \sqrt{5})\right) + 8 = 2(-3 + \sqrt{5}) + 8 = 2 + 2\sqrt{5}$. Points: 1 for using the Fundamental Theorem of Calculus (i.e. $2(f(5) - f(3))$ term), 1 for the final answer $2 + 2\sqrt{5}$.

Solution 3

(c) Mark scheme

- $g(x) = \int_{-2}^{x} f(t)\,dt$, so $g'(x) = f(x)$. $g'(x) = 0$ at $x = -1,\ x = \frac{1}{2},\ x = 5$ (the critical points/endpoints from the given graph of f on $[-2, 5]$). Evaluating g at the candidates: $g(-2) = 0$, $g(-1) = \frac{1}{2}$, $g(\frac{1}{2}) = -\frac{1}{4}$, $g(5) = 11 - \frac{9\pi}{4}$. On $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$. Points: 1 for $g'(x) = f(x)$, 1 for identifying $x = -1$ as a candidate (local max, since f changes from positive to negative there), 1 for the correct answer $g(5) = 11 - \frac{9\pi}{4}$ with justification (candidates test/table).

Solution 3

(d) Mark scheme

- $\lim_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x} = \dfrac{10^1 - 3f'(1)}{f(1) - \arctan 1} = \dfrac{10 - 3\cdot 2}{1 - \arctan 1} = \dfrac{4}{1 - \frac{\pi}{4}}$. Points: 1 for the correct final answer, $\dfrac{4}{1-\pi/4}$.

Question 6

2019 ▪ Q6

Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim\limits_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim\limits_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

Question 6

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

Solution 6

(a)

Mark scheme

- $h'(2) = \dfrac{2}{3}$ (read directly from the given table of values for h and h'). Points: 1 for the correct answer, $h'(2) = 2/3$.

Solution 6

(b)

Mark scheme

- $a(x) = 3x^3 h(x)$. By the product rule, $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$. $a'(2) = 9 \cdot 2^2 \cdot h(2) + 3 \cdot 2^3 \cdot h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 144 + 16 = 160$. Points: 1 for the correct form of the product rule, 1 for the correct expression $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$, 1 for the correct value $a'(2) = 160$.

Solution 6

(c) Mark scheme

- Since h is differentiable, h is continuous, so $\lim_{x\to 2} h(x) = h(2) = 4$. Also $\lim_{x\to 2} h(x) = \lim_{x\to 2} \dfrac{x^2-4}{1-(f(x))^3} = 4$. Since $\lim_{x\to 2}(x^2-4) = 0$ and the overall limit is finite and nonzero, we must have $\lim_{x\to 2}\left(1-(f(x))^3\right) = 0$, so $\lim_{x\to 2} f(x) = 1$. Since f is differentiable (hence continuous), $f(2) = \lim_{x\to 2} f(x) = 1$. Since f is twice differentiable, f' is continuous, so $\lim_{x\to 2} f'(x) = f'(2)$ exists — applying L'Hospital's Rule: $\lim_{x\to 2} \dfrac{x^2-4}{1-(f(x))^3} = \lim_{x\to 2} \dfrac{2x}{-3(f(x))^2 f'(x)} = \dfrac{4}{-3(1)^2 f'(2)} = 4$. Solving gives $-3f'(2) = 4$, so $f'(2) = -\dfrac{1}{3}$. Thus $f(2) = 1$ and $f'(2) = -\dfrac{1}{3}$. Points: 1 for

Solution 6

correctly setting $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$, 1 for finding $f(2) = 1$, 1 for correctly applying L'Hospital's Rule, 1 for the final value $f'(2) = -1/3$.

Solution 6

(d)

Mark scheme

- Because g and h are differentiable (hence continuous), $\lim_{x\to 2} g(x) = g(2) = 4$ and $\lim_{x\to 2} h(x) = h(2) = 4$. Since $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, by the Squeeze Theorem $\lim_{x\to 2} k(x) = 4$. Also $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$. Since $\lim_{x\to 2} k(x) = k(2) = 4$, k is continuous at $x = 2$. Points: 1 for the correct conclusion (continuous at $x = 2$) with justification via the Squeeze Theorem.

Question 5

2018 ▪ Q5

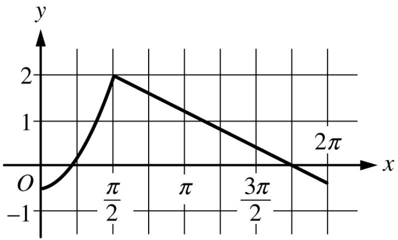
Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \leq x \leq \pi$.

Question 5

Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', th

below. Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify



Graph of g'

Question 5

2018 ▪ Q5

Let f be the function defined by $f(x) = e^x \cos x$.

(b) What is the slope of the line tangent to the graph of f at $x = \dfrac{3\pi}{2}$?

Question 5

2018 ▪ Q5

Let f be the function defined by $f(x) = e^x \cos x$.

(c) Find the absolute minimum value of f on the interval $0 \leq x \leq 2\pi$. Justify your answer.

Question 5

2018 ▪ Q5

Let f be the function defined by $f(x) = e^x \cos x$.

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim\limits_{x\to\pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.

[Graph of g': axes with x-axis labeled at O, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π, and y-axis labeled -1, 1, 2. The curve begins slightly below 0 near $x = 0$, curves upward crossing the x-axis before $x = \frac{\pi}{2}$, rises sharply to a peak at $\left(\frac{\pi}{2}, 2\right)$, then decreases linearly, crossing the x-axis near $x = \frac{3\pi}{2}$, and continues down to a value slightly below 0 at $x = 2\pi$.]

Graph of g'

Solution 5

(a)

Mark scheme

- The average rate of change of f on $0 \leq x \leq \pi$ is $\dfrac{f(\pi) - f(0)}{\pi - 0} = \dfrac{-e^{\pi} - 1}{\pi}$. (1 pt: the answer.)

Solution 5

(b)

Mark scheme

- $f'(x) = e^x \cos x - e^x \sin x$. $f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right) = e^{3\pi/2}$. The slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$ is $e^{3\pi/2}$. (1 pt: correct $f'(x)$; 1 pt: the slope $e^{3\pi/2}$.)

Solution 5

(c) Mark scheme

- $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow x = \frac{\pi}{4}, x = \frac{5\pi}{4}$. Candidates test: $f(0) = 1$, $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}e^{\pi/4}$, $f\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}e^{5\pi/4}$, $f(2\pi) = e^{2\pi}$. The absolute minimum value of f on $0 \leq x \leq 2\pi$ is $-\frac{1}{\sqrt{2}}e^{5\pi/4}$. (1 pt: sets $f'(x) = 0$; 1 pt: identifies $x = \frac{\pi}{4}, \frac{5\pi}{4}$ as candidates; 1 pt: the answer with justification via the candidates test.)

Solution 5

(d) Mark scheme

- $\lim_{x\to\pi/2} f(x) = 0$. Because g is differentiable, g is continuous, so $\lim_{x\to\pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$. By L'Hospital's Rule, $\lim_{x\to\pi/2} \dfrac{f(x)}{g(x)} = \lim_{x\to\pi/2} \dfrac{f'(x)}{g'(x)} = \dfrac{-e^{\pi/2}}{2}$. (1 pt: g continuous at $x = \frac{\pi}{2}$ and both limits equal 0; 1 pt: applies L'Hospital's Rule; 1 pt: the answer $\frac{-e^{\pi/2}}{2}$. Note: max $1/3$ if no limit notation is attached to a ratio of derivatives.)