Past-paper walk-through

Implicit Differentiation ▪ 3.2

eXercise

Questions in this set

- 2025 Q6
- 2024 Q5
- 2023 Q6
- 2021 Q5
- 2015 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2025 ▪ Q6

Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{-x}{2(3y^2 - 2y - 1)}$.

Question 6

2025 ▪ Q6

Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

(b) There is a point P on the curve G near $(2, -1)$ with x-coordinate 1.6. Use the line tangent to the curve at $(2, -1)$ to approximate the y-coordinate of point P.

Question 6

2025 ▪ Q6

Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

(c) For $x > 0$ and $y > 0$, there is a point S on the curve G at which the line tangent to the curve at that point is vertical. Find the y-coordinate of point S. Show the work that leads to your answer.

Question 6

2025 ▪ Q6

Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

(d) A particle moves along the curve H defined by the equation $2xy + \ln y = 8$. At the instant when the particle is at the point $(4, 1)$, $\frac{dx}{dt} = 3$. Find $\frac{dy}{dt}$ at that instant. Show the work that leads to your answer.

Solution 6

(a) Mark scheme

- Differentiate $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$ implicitly with respect to x:
  $3y^2\frac{dy}{dx} - 2y\frac{dy}{dx} - \frac{dy}{dx} + \frac{x}{2} = 0 \Rightarrow \left(3y^2 - 2y - 1\right)\frac{dy}{dx} = -\frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{-x}{2(3y^2 - 2y - 1)}$.
  Point 1: correct implicit differentiation of the given equation (may use y' in place of $\frac{dy}{dx}$). Point 2: correctly verifies/derives the given result $\frac{dy}{dx} = \frac{-x}{2(3y^2 - 2y - 1)}$ (it suffices to present $(3y^2 - 2y - 1)\frac{dy}{dx} = -\frac{x}{2}$ with no subsequent errors); requires Point 1.

Solution 6

(b) Mark scheme

- Slope of the tangent line at $(2, -1)$: $\left.\dfrac{dy}{dx}\right|_{(2,-1)} = \dfrac{-2}{2(3(-1)^2 - 2(-1) - 1)} = \dfrac{-2}{2(3+2-1)} = \dfrac{-2}{8} = -\dfrac{1}{4}$. Tangent line approximation: $y \approx -1 + \left(-\dfrac{1}{4}\right)(1.6-2) = -1 - \dfrac{1}{4}(-0.4) = -1 + 0.1 = -0.9$. Point 3: correct slope of the tangent line at $(2, -1)$, i.e. $\left.\dfrac{dy}{dx}\right|_{(x,y)=(2,-1)} = -\dfrac{1}{4}$ (or equivalent statement, e.g. "slope is $-\dfrac{1}{4}$"). Point 4: correct tangent-line approximation $y \approx -0.9$, using $x = 1.6$; requires clearly demonstrated use of the

Solution 6

tangent line at $x = 1.6$ (does not require the tangent line equation to be explicitly presented).

Solution 6

(c) Mark scheme

- The curve has a vertical tangent where the denominator of $\dfrac{dy}{dx}$ is zero: $2(3y^2 - 2y - 1) = 0 \Rightarrow 2(3y + 1)(y - 1) = 0 \Rightarrow y = -\dfrac{1}{3}$ or $y = 1$. Since $y > 0$ is required, $y = 1$ is the y-coordinate of point S. Point 5: sets the denominator $2(3y^2 - 2y - 1) = 0$ (or an equivalent factored/unfactored form, e.g. $3y^2 - 2y - 1 = 0$ or $(3y \pm 1)(y \pm 1) = 0$; the numerator need not be considered). Point 6: correct answer $y = 1$, with correct algebraic work clearly identifying it as the only solution satisfying $y > 0$ (stating both $y = -\frac{1}{3}$ and $y = 1$ without identifying $y = 1$ as the answer to the prompt does not earn this point); requires Point 5.

Solution 6

(d) Mark scheme

- Differentiate $2xy + \ln y = 8$ implicitly with respect to t: $2\dfrac{dx}{dt}y + 2x\dfrac{dy}{dt} + \dfrac{1}{y}\dfrac{dy}{dt} = 0.$

  At $(x, y) = (4, 1)$ with $\dfrac{dx}{dt} = 3$:
  $2(3)(1) + 2(4)\dfrac{dy}{dt} + \dfrac{1}{1}\dfrac{dy}{dt} = 0 \Rightarrow 6 + 9\dfrac{dy}{dt} = 0 \Rightarrow \dfrac{dy}{dt} = -\dfrac{2}{3}$. Point 7: attempts implicit differentiation of $2xy + \ln y = 8$ with respect to t (at most one error; alternatively, differentiating with respect to x first is acceptable as long as $\dfrac{dy}{dx}$ is eventually correctly linked to $\dfrac{dy}{dt}$ via the chain rule). Point 8: correct equation

Solution 6

$2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$ (or equivalent); requires Point 7 with no errors in the implicit differentiation. Point 9: correct answer $\frac{dy}{dt} = -\frac{2}{3}$; requires Points 7 and 8.

Question 5

2024 ▪ Q5

Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\dfrac{dy}{dx} = \dfrac{-2x}{3+4y}$.

(a) There is a point on the curve near $(2, 4)$ with x-coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y-coordinate of this point.

(b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

(c) The curve intersects the positive x-axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

(d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y-coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x-coordinate of the particle's position with respect to time?

Solution 5

(a) Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{(2,4)} = \dfrac{-2(2)}{3+4(4)} = -\dfrac{4}{19}$. Tangent line approximation: $y \approx 4 - \dfrac{4}{19}(3-2) = \dfrac{72}{19}$. 1 point for the correct slope $-4/19$ at $(2,4)$, 1 point for the correct approximation using that slope evaluated at $x = 3$ through the point $(2,4)$.

Solution 5

(b) Mark scheme

- $\frac{dy}{dx} = \frac{-2x}{3+4y} = 0 \Rightarrow x = 0$, so if $y = 1$ were tangent, the point of tangency would have to be $(0,1)$. But $(0,1)$ is not on the curve, because $0^2 + 3(1) + 2(1)^2 = 5 \neq 48$. Therefore the horizontal line $y = 1$ is NOT tangent to the curve. 1 point for considering $dy/dx = 0$ (equivalently $x = 0$, or identifying the point $(0,1)$), 1 point for the answer 'not tangent' with a valid reason (merely stating '(0,1) is not on the curve' without explanation is insufficient; must show why).

Solution 5

(c) Mark scheme

- At $(\sqrt{48}, 0)$, $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3+4(0)} = \frac{-2\sqrt{48}}{3}$. Since the denominator $3 + 4(0) = 3 \neq 0$, the slope is defined (finite), so the tangent line at this point is NOT vertical. The single point requires clearly showing the slope of the tangent at $(\sqrt{48}, 0)$ is defined and answering 'no' (considering only that the denominator is nonzero is sufficient).

Solution 5

(d) Mark scheme

- Curve: $y^3 + 2xy = 24$. Implicit differentiation with respect to t:
  $3y^2\dfrac{dy}{dt} + 2x\dfrac{dy}{dt} + 2y\dfrac{dx}{dt} = 0$. At $(4, 2)$ with $\dfrac{dy}{dt} = -2$:
  $3(2)^2(-2) + 2(4)(-2) + 2(2)\dfrac{dx}{dt} = 0 \Rightarrow -24 - 16 + 4\dfrac{dx}{dt} = 0 \Rightarrow \dfrac{dx}{dt} = \dfrac{40}{4} = 10.$
  The rate of change of the x-coordinate is 10 units per second. 1 point for attempting implicit differentiation of $y^3 + 2xy = 24$ with respect to t (at most one error), 1 point for the fully correct differentiated equation $3y^2\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$, 1 point for correctly using $dy/dt = -2$ (the y-coordinate is decreasing), 1 point (only earnable with the prior three, no arithmetic mistakes) for the final answer 10.

Question 6

2023 ▪ Q6

Consider the curve given by the equation $6xy = 2 + y^3$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2y}{y^2 - 2x}$.

Question 6

2023 ▪ Q6

Consider the curve given by the equation $6xy = 2 + y^3$.
(b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.

Question 6

2023 ▪ Q6

Consider the curve given by the equation $6xy = 2 + y^3$.

(c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

Question 6

2023 ▪ Q6

Consider the curve given by the equation $6xy = 2 + y^3$.

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$ unit per second. What is the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

Solution 6

(a) Mark scheme

- Differentiate implicitly: $\frac{d}{dx}(6xy) = \frac{d}{dx}(2 + y^3) \Rightarrow 6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$ (1 pt: correct implicit differentiation of $6xy = 2 + y^3$). Solve for dy/dx: $2y = \frac{dy}{dx}(y^2 - 2x) \Rightarrow \frac{dy}{dx} = \frac{2y}{y^2 - 2x}$ (1 pt: verification, i.e. correctly isolating dy/dx to match the given expression, with no subsequent errors).

Solution 6

(b)

Mark scheme

- For the tangent line to be horizontal, it is necessary that $2y = 0$ (so y=0) and $y^2 - 2x \neq 0$ (1 pt: sets 2y=0, equivalently y=0, dy/dx=0, or dy=0). Substituting y=0 into $6xy = 2 + y^3$ yields $6x \cdot 0 = 2$, i.e. $0 = 2$, which has no solution; therefore there is no point on the curve at which the tangent line is horizontal (1 pt: answer with reason).

Solution 6

(c)

Mark scheme

- For the tangent line to be vertical, it is necessary that $2y \neq 0$ and $y^2 - 2x = 0$ (so $x = y^2/2$, equivalently $y = \sqrt{2x}$) (1 pt: sets $y^2 - 2x = 0$). Substituting $x = y^2/2$ into $6xy = 2 + y^3$: $3y^2 \cdot y = 2 + y^3 \Rightarrow 2y^3 = 2 \Rightarrow y = 1$ (1 pt: substitutes correctly and solves). Substituting y=1 into $6xy = 2 + y^3$ gives $6x = 3$, so $x = \frac{1}{2}$; the tangent line is vertical at the point $\left(\frac{1}{2}, 1\right)$ (1 pt: answer — both coordinates must be presented, and the point verified to lie on the curve).

Solution 6

(d) Mark scheme

- Differentiate $6xy = 2 + y^3$ implicitly with respect to t: $6y\dfrac{dx}{dt} + 6x\dfrac{dy}{dt} = 3y^2\dfrac{dy}{dt}$ (1 pt: implicit differentiation with respect to t, presenting at least one of the terms $6y\,dx/dt$, $6x\,dy/dt$, or $3y^2\,dy/dt$). At the point $(x, y) = \left(\dfrac{1}{2}, -2\right)$ with $\dfrac{dx}{dt} = \dfrac{2}{3}$: $6(-2)\left(\dfrac{2}{3}\right) + 6\left(\dfrac{1}{2}\right)\dfrac{dy}{dt} = 3(-2)^2\dfrac{dy}{dt} \Rightarrow -8 + 3\dfrac{dy}{dt} = 12\dfrac{dy}{dt} \Rightarrow \dfrac{dy}{dt} = -\dfrac{8}{9}$ unit per second (1 pt: answer, an unsupported value of -8/9 alone earns no points).

Question 5

2021 ▪ Q5

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y\sin x$ for $y > 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y\cos x}{4y - \sin x}$.

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

(c) For $0 \leq x \leq \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

Solution 5

(a) Mark scheme

- Implicitly differentiate $2y^2 - 6 = y \sin x$: $\dfrac{d}{dx}(2y^2 - 6) = \dfrac{d}{dx}(y \sin x) \Rightarrow 4y\dfrac{dy}{dx} = \dfrac{dy}{dx}\sin x + y\cos x$. Then $4y\dfrac{dy}{dx} - \dfrac{dy}{dx}\sin x = y\cos x \Rightarrow \dfrac{dy}{dx}(4y - \sin x) = y\cos x \Rightarrow \dfrac{dy}{dx} = \dfrac{y\cos x}{4y - \sin x}$, as required. Point 1: correct implicit differentiation of $2y^2 - 6 = y\sin x$ (alternative notations for dy/dx such as y' are fine) — required for point 2. Point 2: verification, i.e. reaching $\frac{dy}{dx}(4y - \sin x) = y\cos x$ is sufficient with no subsequent errors.

Solution 5

(b)

Mark scheme

- At the point $(0, \sqrt{3})$: $\dfrac{dy}{dx} = \dfrac{\sqrt{3}\cos 0}{4\sqrt{3} - \sin 0} = \dfrac{1}{4}$. An equation for the tangent line is $y = \sqrt{3} + \dfrac{1}{4}x$. Point 1: any correct tangent line equation; no supporting work is required and the slope value need not be simplified.

Solution 5

(c) Mark scheme

- $\dfrac{dy}{dx} = \dfrac{y\cos x}{4y - \sin x} = 0 \Rightarrow y\cos x = 0$ and $4y - \sin x \neq 0$. Since $y\cos x = 0$ and $y > 0 \Rightarrow \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}$ (on $0 \leq x \leq \pi$). When $x = \pi/2$: $y\sin(\pi/2) = 2y^2 - 6 \Rightarrow y = 2y^2 - 6 \Rightarrow 2y^2 - y - 6 = 0 \Rightarrow (2y+3)(y-2) = 0 \Rightarrow y = 2$ (since $y > 0$). Check: $4y - \sin x = 8 - 1 = 7 \neq 0$. The tangent line is horizontal at $\left(\dfrac{\pi}{2}, 2\right)$. Point 1: sets $\frac{dy}{dx} = 0$ (or equivalently $y\cos x = 0$ or $\cos x = 0$). Point 2: commits to $x = \pi/2$ only (any extraneous additional x-values forfeit this point; y-values presented are not considered here). Point 3: $y = 2$ (not earned without point 2; coordinates need not be presented as an ordered pair).

Solution 5

(d) Mark scheme

- $\dfrac{d^2y}{dx^2} = \dfrac{(4y - \sin x)\left(\frac{dy}{dx}\cos x - y\sin x\right) - (y\cos x)\left(4\frac{dy}{dx} - \cos x\right)}{(4y - \sin x)^2}$. At $x = \pi/2, y = 2$ (with $dy/dx = 0$): $\dfrac{d^2y}{dx^2} = \dfrac{(7)(-2) - (0)(0)}{49} = \dfrac{-2}{7} < 0$. Since $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$ at $\left(\frac{\pi}{2}, 2\right)$, f has a RELATIVE MAXIMUM there (Second Derivative Test). [Alternate solution via First Derivative Test: near $(\pi/2, 2)$, $4y - \sin x > 0$ and $y > 0$, so $\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$ changes from positive to negative at $x = \pi/2$, giving a relative maximum by the First Derivative Test.] Point 1: attempt to use the quotient (or product) rule to find d^2y/dx^2 [or: considers the

Solution 5

sign of $4y - \sin x$]. Point 2: correctly finds d^2y/dx^2 and evaluates it as negative at the point [or: shows dy/dx changes sign $+$ to $-$ at $x = \pi/2$ — can be earned without point 1]. Point 3: correct conclusion (relative maximum) with justification, consistent with a reported nonzero value/sign obtained using $dy/dx = 0$; a response concluding "minimum" earns no point-3 credit.

Question 6

2015 ▪ Q6

Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\dfrac{dy}{dx} = \dfrac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

(c) Evaluate $\dfrac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

Solution 6

(a)

Mark scheme

- dy/dx at $(x, y) = (-1, 1)$ is $1 / (3(1)^2 - (-1)) = 1/4$. Equation of the tangent line: $y = (1/4)(x + 1) + 1$. Points: 1 for the correct slope $1/4$, 1 for the correct tangent line equation.

Solution 6

(b)

Mark scheme

- The tangent line is vertical where the denominator $3y^2 - x = 0$, i.e. $x = 3y^2$. Substituting into $y^3 - xy = 2$: $y^3 - (3y^2)(y) = 2 \quad -2y^3 = 2 \quad y^3 = -1 \quad y = -1$. Then $x = 3y^2 = 3(-1)^2 = 3$ (consistent with the original equation: $(-1)^3 - x(-1) = 2 \quad -1 + x = 2 \quad x = 3$). The tangent line to the curve is vertical at the point $(3, -1)$. Points: 1 for setting $3y^2 - x = 0$, 1 for reducing to an equation in one variable (e.g. $-2y^3 = 2$), 1 for the correct coordinates $(3, -1)$.

Solution 6

(c) Mark scheme

- Differentiating $dy/dx = y/(3y^2 - x)$ implicitly with respect to x gives $d^2y/dx^2 = [(3y^2 - x)(dy/dx) - y(6y(dy/dx) - 1)] / (3y^2 - x)^2$. At $(x, y) = (-1, 1)$ with $dy/dx = 1/4$: $3y^2 - x = 3(1)^2 - (-1) = 4$, so $(3y^2 - x)(dy/dx) = 4 \cdot (1/4) = 1$; $6y(dy/dx) - 1 = 6(1)(1/4) - 1 = 3/2 - 1 = 1/2$, so $y \cdot (6y(dy/dx) - 1) = 1 \cdot (1/2) = 1/2$. Numerator $= 1 - 1/2 = 1/2$. Denominator $= 4^2 = 16$. So $d^2y/dx^2|\_(-1,1) = (1/2)/16 = 1/32$. Points: 2 for correct implicit differentiation to obtain d^2y/dx^2 in terms of x, y, and dy/dx, 1 for substituting $dy/dx = 1/4$ (and $x = -1$, $y = 1$) into that expression, 1 for the correct final numeric answer $1/32$.