Past-paper walk-through

The Product Rule ▪ 2.8

eXercise

Questions in this set

- 2026 Q6
- 2023 Q5
- 2021 Q4
- 2019 Q6
- 2018 Q5
- 2017 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2026 ▪ Q6

The function f is twice differentiable. The table gives values of f and its derivative f' at selected values of x.

x	**0**	**2**	**3**	**6**
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Question 6

(a) Find $\lim_{x \to 2} \frac{f(x)}{x}$, or state that the limit does not exist.

(b) Let $g(x) = f(f(x))$. Find $g'(2)$. Show the work that leads to your answer.

(c) Let h be a differentiable function such that $h(0) = 10$ and $h'(x) = f'(3x)$. Find $h(2)$. Show the work that leads to your answer.

(d)(i) Let k be the function defined by $k(x) = \int_0^x t^2 f(t)\, dt$. Find $k'(x)$.

(d)(ii) Find $k''(3)$. Show the work that leads to your answer.

Question 6

x	0	2	3	6
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Question 5

2023 ▪ Q5

The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

(a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

(b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.

(c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

Question 5

(d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$? Justify your answer.

Question 5

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

Solution 5

(a)

Mark scheme

- $h'(x) = f'(g(x)) \cdot g'(x)$ (1 pt: chain rule).
  $h'(7) = f'(g(7)) \cdot g'(7) = f'(0) \cdot 8 = \frac{3}{2} \cdot 8 = 12$ (1 pt: correct answer 12, using the table values g(7)=0, f'(0)=3/2, g'(7)=8).

Solution 5

(b)

Mark scheme

- $k''(x) = 2f(x)f'(x)g(x) + (f(x))^2g'(x)$ (1 pt: product/chain rule applied to k'(x)=(f(x))^2 g(x)). $k''(4) = 2f(4)f'(4)g(4) + (f(4))^2g'(4) = 2(4)(3)(-3) + (4)^2(2) = -72 + 32 = -40$ (1 pt: k"(4)=-40 with supporting work using the table values f(4)=4, f'(4)=3, g(4)=-3, g'(4)=2). Since k"(4)=-40<0 and k" is continuous, the graph of k is concave down at x=4 (1 pt: answer with reason, consistent with the declared nonzero value of k"(4)).

Solution 5

(c)

Mark scheme

- By the Fundamental Theorem of Calculus, $m(2) = 5(2)^3 + \int_0^2 f'(t)\,dt = 40 + (f(2) - f(0)) = 40 + (7 - 10) = 37$ (1 pt: answer 37 with supporting work equivalent to $5 \cdot 8 + (f(2) - f(0))$).

Solution 5

(d)

Mark scheme

- $m'(x) = 15x^2 + f'(x)$ (1 pt: considers m'(x)).
  $m'(2) = 15(2)^2 + f'(2) = 60 + (-8) = 52$ (1 pt: m'(2) with supporting work, using f'(2)=-8 from the table; an unsupported m'(2)=52 does not earn this point). Since $m'(2) = 52 > 0$, the graph of m is increasing at x=2 (1 pt: answer with justification, consistent with the declared value of m'(2)).

Question 4

2021 ▪ Q4

[Graph of f on a grid, x-axis from -4 to 6, y-axis from -4 to 6, gridlines every 2 units on the y-axis and every 2 units on the x-axis; the graph consists of four connected line segments: from $(-4, 0)$ up to $(-2, 6)$, then down to $(0, 4)$, then down to $(2, -4)$, then up to $(6, 0)$. Labeled "Graph of f".]

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \displaystyle\int_0^x f(t)\, dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

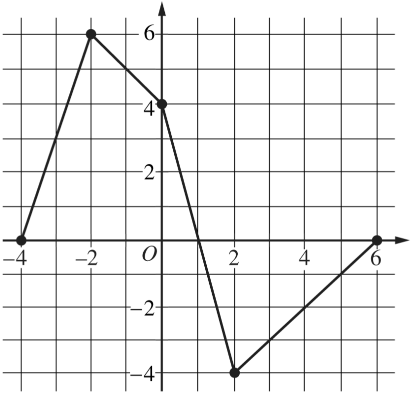
(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

Question 4

(c) Find $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- $G'(x) = f(x)$ (the "global point," worth 1 point, earnable anywhere in parts (a)-(c) by writing $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$, $G'(3) = f(3)$, or $G'(2) = f(2)$). The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals (intervals may include one or both endpoints). Point (part a, 1 point): correct intervals with the reason that $G' = f$ is increasing there. Point (global, 1 point, folded into this leaf): correctly stating $G'(x) = f(x)$ anywhere in the response.

Solution 4

(b)

Mark scheme

- $P(x) = G(x)f(x) \Rightarrow P'(x) = G(x)f'(x) + f(x)G'(x)$. Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ (and $f'(3) = 1$, the slope of the line segment from $(2, -4)$ to $(6, 0)$): $P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$. Point 1: correct application of the product rule in terms of x or evaluated at 3 (once earned, cannot be lost). Point 2: correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$. Point 3: the final answer 5.5 (needs points 1 and 2 first; simplification not required).

Solution 4

(c) Mark scheme

- $\lim_{x\to 2}(x^2 - 2x) = 0$ and, since G is continuous, $\lim_{x\to 2} G(x) = \int_0^2 f(t)\,dt = 0$, so the limit is the indeterminate form $0/0$. By L'Hospital's Rule, $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x} = \lim_{x\to 2} \dfrac{G'(x)}{2x-2} = \lim_{x\to 2} \dfrac{f(x)}{2x-2} = \dfrac{f(2)}{2} = \dfrac{-4}{2} = -2$. Point 1: shows $\lim(x^2 - 2x) = 0$ and $\lim G(x) = 0$, and shows the ratio of the two derivatives $G'(x)$ and $2x - 2$ (may be shown as an evaluation at $x = 2$). Point 2: evaluates the limit correctly with appropriate limit notation, including $\lim_{x\to 2} \frac{G'(x)}{2x-2}$ or $\lim_{x\to 2} \frac{f(x)}{2x-2}$; any linkage error (e.g. writing $\frac{G'(x)}{2x-2} = \frac{f(2)}{2}$ without the limit) forfeits this point.

Solution 4

(d) Mark scheme

- $G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$. Average rate of change $= \dfrac{G(2) - G(-4)}{2-(-4)} = \dfrac{0-(-16)}{6} = \dfrac{8}{3}$. Yes — since $G'(x) = f(x)$, G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$, so the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \dfrac{8}{3}$. Point 1: presents at least a difference and a quotient with correct evaluation (numerical simplification not required, but if simplified must be correct). Point 2: correct conclusion (earnable even with an incorrect/no evaluation of the average rate of change, as long as MVT's conditions and conclusion are stated — an explicit MVT citation is not required).

Solution 4

Question 6

2019 ▪ Q6

Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

Question 6

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

Solution 6

(a)

Mark scheme

- $h'(2) = \dfrac{2}{3}$ (read directly from the given table of values for h and h'). Points: 1 for the correct answer, $h'(2) = 2/3$.

Solution 6

(b)

Mark scheme

- $a(x) = 3x^3 h(x)$. By the product rule, $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$. $a'(2) = 9 \cdot 2^2 \cdot h(2) + 3 \cdot 2^3 \cdot h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 144 + 16 = 160$. Points: 1 for the correct form of the product rule, 1 for the correct expression $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$, 1 for the correct value $a'(2) = 160$.

Solution 6

(c) Mark scheme

- Since h is differentiable, h is continuous, so $\lim_{x\to 2} h(x) = h(2) = 4$. Also $\lim_{x\to 2} h(x) = \lim_{x\to 2} \dfrac{x^2-4}{1-(f(x))^3} = 4$. Since $\lim_{x\to 2}(x^2-4) = 0$ and the overall limit is finite and nonzero, we must have $\lim_{x\to 2}\left(1-(f(x))^3\right) = 0$, so $\lim_{x\to 2} f(x) = 1$. Since f is differentiable (hence continuous), $f(2) = \lim_{x\to 2} f(x) = 1$. Since f is twice differentiable, f' is continuous, so $\lim_{x\to 2} f'(x) = f'(2)$ exists — applying L'Hospital's Rule: $\lim_{x\to 2} \dfrac{x^2-4}{1-(f(x))^3} = \lim_{x\to 2} \dfrac{2x}{-3(f(x))^2 f'(x)} = \dfrac{4}{-3(1)^2 f'(2)} = 4$. Solving gives $-3f'(2) = 4$, so $f'(2) = -\dfrac{1}{3}$. Thus $f(2) = 1$ and $f'(2) = -\dfrac{1}{3}$. Points: 1 for

Solution 6

correctly setting $\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = 4$, 1 for finding $f(2) = 1$, 1 for correctly applying L'Hospital's Rule, 1 for the final value $f'(2) = -1/3$.

Solution 6

(d)

Mark scheme

- Because g and h are differentiable (hence continuous), $\lim_{x\to 2} g(x) = g(2) = 4$ and $\lim_{x\to 2} h(x) = h(2) = 4$. Since $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$, by the Squeeze Theorem $\lim_{x\to 2} k(x) = 4$. Also $4 = g(2) \leq k(2) \leq h(2) = 4$, so $k(2) = 4$. Since $\lim_{x\to 2} k(x) = k(2) = 4$, k is continuous at $x = 2$. Points: 1 for the correct conclusion (continuous at $x = 2$) with justification via the Squeeze Theorem.

Question 5

2018 ▪ Q5

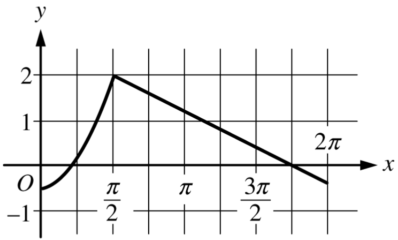
Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \leq x \leq \pi$.

Question 5

Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', th

below. Find the value of $\lim\limits_{x \to \pi/2} \dfrac{f(x)}{g(x)}$ or state that it does not exist. Justify



Graph of g'

Question 5

2018 ▪ Q5

Let f be the function defined by $f(x) = e^x \cos x$.

(b) What is the slope of the line tangent to the graph of f at $x = \dfrac{3\pi}{2}$?

Question 5

2018 ▪ Q5

Let f be the function defined by $f(x) = e^x \cos x$.

(c) Find the absolute minimum value of f on the interval $0 \leq x \leq 2\pi$. Justify your answer.

Question 5

2018 ▪ Q5

Let f be the function defined by $f(x) = e^x \cos x$.

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x\to\pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.

[Graph of g': axes with x-axis labeled at O, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π, and y-axis labeled -1, 1, 2. The curve begins slightly below 0 near $x = 0$, curves upward crossing the x-axis before $x = \frac{\pi}{2}$, rises sharply to a peak at $\left(\frac{\pi}{2}, 2\right)$, then decreases linearly, crossing the x-axis near $x = \frac{3\pi}{2}$, and continues down to a value slightly below 0 at $x = 2\pi$.]

Graph of g'

Solution 5

(a)

Mark scheme

- The average rate of change of f on $0 \leq x \leq \pi$ is $\dfrac{f(\pi) - f(0)}{\pi - 0} = \dfrac{-e^{\pi} - 1}{\pi}$. (1 pt: the answer.)

Solution 5

(b)

Mark scheme

- $f'(x) = e^x \cos x - e^x \sin x$. $f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right) = e^{3\pi/2}$. The slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$ is $e^{3\pi/2}$. (1 pt: correct $f'(x)$; 1 pt: the slope $e^{3\pi/2}$.)

Solution 5

(c) Mark scheme

- $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow x = \frac{\pi}{4}, x = \frac{5\pi}{4}$. Candidates test: $f(0) = 1$, $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}e^{\pi/4}$, $f\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}e^{5\pi/4}$, $f(2\pi) = e^{2\pi}$. The absolute minimum value of f on $0 \leq x \leq 2\pi$ is $-\frac{1}{\sqrt{2}}e^{5\pi/4}$. (1 pt: sets $f'(x) = 0$; 1 pt: identifies $x = \frac{\pi}{4}, \frac{5\pi}{4}$ as candidates; 1 pt: the answer with justification via the candidates test.)

Solution 5

(d) Mark scheme

- $\lim_{x\to\pi/2} f(x) = 0$. Because g is differentiable, g is continuous, so $\lim_{x\to\pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$. By L'Hospital's Rule, $\lim_{x\to\pi/2} \dfrac{f(x)}{g(x)} = \lim_{x\to\pi/2} \dfrac{f'(x)}{g'(x)} = \dfrac{-e^{\pi/2}}{2}$. (1 pt: g continuous at $x = \frac{\pi}{2}$ and both limits equal 0; 1 pt: applies L'Hospital's Rule; 1 pt: the answer $\frac{-e^{\pi/2}}{2}$. Note: max $1/3$ if no limit notation is attached to a ratio of derivatives.)

Question 6

2017 ▪ Q6

Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x.

x	$g(x)$	$g'(x)$
-5	10	-3
-4	5	-1
-3	2	4
-2	3	1
-1	1	-2
0	0	-3

Question 6

Let h be the function whose graph, consisting of five line segments, is shown in the figure above.
[Graph of h, labeled "Graph of h", on a gridded coordinate plane with axes x and y; gridlines at each integer, with 1 marked on both axes near the origin. The graph consists of five line segments: starting at approximately $(-5, 3)$, rising to a flat peak spanning roughly $(-3.5, 4)$ to $(-3, 4)$, then decreasing linearly, crossing the x-axis slightly to the right of the y-axis (around $x = 0.5$), continuing to decrease to a local minimum at approximately $(2, -1)$, then rising steeply and linearly to approximately $(4, 5)$.]

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

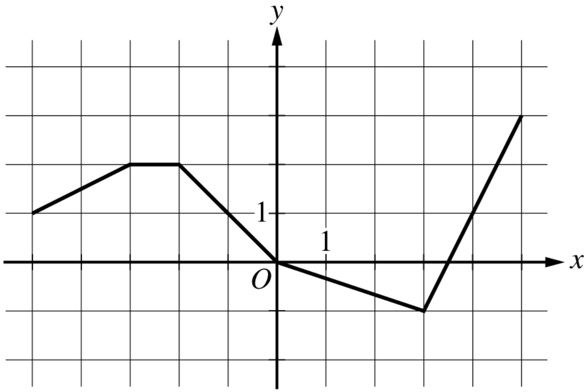
(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

Question 6

(d) Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -4$? Justify your answer.

Question 6

x	$g(x)$	$g'(x)$
–5	10	–3
–4	5	–1
–3	2	4
–2	3	1
–1	1	–2
0	0	–3



Graph of h