Past-paper walk-through

Defining Average and Instantaneous Rates of Change at a Point ▪ 2.1

eXercise

Questions in this set

- 2024 Q1
- 2022 Q4
- 2018 Q4
- 2018 Q5
- 2015 Q2
- 2014 Q1
- 2014 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \leq t \leq 7$. Show the work that leads to your answer and include units of measure.

Question 1

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(c) For $12 \leq t \leq 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \dfrac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Solution 1

(a)

Mark scheme

- $C'(5) \approx \dfrac{C(7) - C(3)}{7 - 3} = \dfrac{69 - 85}{4} = -4$ degrees Celsius per minute. 1 point for the difference-quotient setup with supporting work (a difference over a quotient using table values), 1 point for the correct value with units ('degrees Celsius per minute' attached to the numeric value).

Solution 1

(b) Mark scheme

- Left Riemann sum with the three subintervals from the table: $\int_0^{12} C(t)\,dt \approx (3-0)\cdot C(0) + (7-3)\cdot C(3) + (12-7)\cdot C(7) = 3(100) + 4(85) + 5(69) = 300 + 340 + 345 = 985$. Interpretation: $\frac{1}{12}\int_0^{12} C(t)\,dt$ is the average temperature of the coffee (in degrees Celsius) over the interval $t = 0$ to $t = 12$. 1 point for the correct form of the left Riemann sum (widths times left-endpoint values), 1 point for the numeric estimate 985, 1 point for an interpretation that names both 'average temperature' and the time interval.

Solution 1

(c)

Mark scheme

- $C(20) = C(12) + \int_{12}^{20} C'(t)\,dt = 55 - 14.670812 = 40.329188 \approx 40.329$ degrees Celsius. 1 point for setting up the definite integral of $C'(t)$ with correct limits 12 to 20, 1 point for adding the initial condition $C(12) = 55$ (symbolically or numerically), 1 point for the final numeric answer 40.329 (or 40.330) with supporting work shown.

Solution 1

(d) Mark scheme

- Because $C''(t) > 0$ on $12 < t < 20$ (from $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$, numerator and denominator both positive on this interval), the rate of change of temperature $C'(t)$ is increasing on this interval — that is, the temperature of the coffee is changing at an increasing rate. The single point requires a correct answer ('increasing rate') together with a reason that references the sign of the second derivative of C (not just evaluation at one point, and not an ambiguous pronoun).

Question 4

2022 ▪ Q4

| t (days) | 0 | 3 | 7 | 10 | 12 | |—|—|—|—|—|—| | $r'(t)$ (centimeters per day) | -6.1 | -5.0 | -4.4 | -3.8 | -3.5 |

An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

Question 4

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of

$$\int_0^{12} r'(t)\,dt.$$

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Question 4

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

Solution 4

(a)

Mark scheme

- $r''(8.5) \approx \dfrac{r'(10) - r'(7)}{10 - 7} = \dfrac{-3.8 - (-4.4)}{3} = \dfrac{0.6}{3} = 0.2$ centimeters per day per day. Points: (1) correct difference-quotient setup (a difference over a quotient) with supporting work; (2) correct units (cm/day^2, or an equivalent form).

Solution 4

(b)

Mark scheme

- r is twice differentiable, so r' is differentiable and therefore continuous. $r'(0) = -6.1 < -6 < -5.0 = r'(3)$. By the Intermediate Value Theorem, there is a time t, $0 < t < 3$, such that $r'(t) = -6$. Points: (1) establishes that -6 is between $r'(0) = -6.1$ and $r'(3) = -5.0$; (2) correct conclusion invoking the Intermediate Value Theorem (or equivalent reasoning), noting that r' is continuous because r is twice differentiable.

Solution 4

(c)

Mark scheme

- $\int_0^{12} r'(t)\,dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12) = 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5) = -51$. Points: (1) correct form of the right Riemann sum (subinterval widths 3, 4, 3, 2 paired with the right endpoints $r'(3), r'(7), r'(10), r'(12)$); (2) correct answer, -51.

Solution 4

(d) Mark scheme

- $V = \frac{1}{3}\pi r^2 h \Rightarrow \dfrac{dV}{dt} = \dfrac{2}{3}\pi r h \dfrac{dr}{dt} + \dfrac{1}{3}\pi r^2 \dfrac{dh}{dt}$. At $t = 3$: $r = 100$, $h = 50$, $\frac{dr}{dt} = r'(3) = -5$ (from the table in part (a)), $\frac{dh}{dt} = -2$ (height decreases at 2 cm/day). $\dfrac{dV}{dt}\Big|_{t=3} = \dfrac{2}{3}\pi(100)(50)(-5) + \dfrac{1}{3}\pi(100)^2(-2) = -\dfrac{70{,}000\pi}{3}$ cubic centimeters per day. Points: (1) correct product-rule term; (2) correct chain-rule term (both differentials present); (3) correct final answer.

Question 4

2018 ▪ Q4

t **(years)**	**2**	**3**	**5**	**7**	**10**
$H(t)$ (meters)	1.5	2	6	11	15

The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

Question 4

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

Question 4

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15