Past-paper walk-through

Determining Limits Using the Squeeze Theorem ▪ 1.8

eXercise

Questions in this set

- 2019 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2019 ▪ Q6

Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

Question 6

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

Solution 6

(a)

Mark scheme

- $h'(2) = \dfrac{2}{3}$ (read directly from the given table of values for h and h'). Points: 1 for the correct answer, $h'(2) = 2/3$.

Solution 6

(b)

Mark scheme

- $a(x) = 3x^3 h(x)$. By the product rule, $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$. $a'(2) = 9 \cdot 2^2 \cdot h(2) + 3 \cdot 2^3 \cdot h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 144 + 16 = 160$. Points: 1 for the correct form of the product rule, 1 for the correct expression $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$, 1 for the correct value $a'(2) = 160$.

Solution 6

(c) Mark scheme

- Since h is differentiable, h is continuous, so $\lim_{x\to 2} h(x) = h(2) = 4$. Also $\lim_{x\to 2} h(x) = \lim_{x\to 2} \dfrac{x^2-4}{1-(f(x))^3} = 4$. Since $\lim_{x\to 2}(x^2-4) = 0$ and the overall limit is finite and nonzero, we must have $\lim_{x\to 2}\left(1-(f(x))^3\right) = 0$, so $\lim_{x\to 2} f(x) = 1$. Since f is differentiable (hence continuous), $f(2) = \lim_{x\to 2} f(x) = 1$. Since f is twice differentiable, f' is continuous, so $\lim_{x\to 2} f'(x) = f'(2)$ exists — applying L'Hospital's Rule: $\lim_{x\to 2} \dfrac{x^2-4}{1-(f(x))^3} = \lim_{x\to 2} \dfrac{2x}{-3(f(x))^2 f'(x)} = \dfrac{4}{-3(1)^2 f'(2)} = 4$. Solving gives $-3f'(2) = 4$, so $f'(2) = -\dfrac{1}{3}$. Thus $f(2) = 1$ and $f'(2) = -\dfrac{1}{3}$. Points: 1 for

Solution 6

correctly setting $\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = 4$, 1 for finding $f(2) = 1$, 1 for correctly applying L'Hospital's Rule, 1 for the final value $f'(2) = -1/3$.

Solution 6

(d)

Mark scheme

- Because g and h are differentiable (hence continuous), $\lim_{x\to 2} g(x) = g(2) = 4$ and $\lim_{x\to 2} h(x) = h(2) = 4$. Since $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, by the Squeeze Theorem $\lim_{x\to 2} k(x) = 4$. Also $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$. Since $\lim_{x\to 2} k(x) = k(2) = 4$, k is continuous at $x = 2$. Points: 1 for the correct conclusion (continuous at $x = 2$) with justification via the Squeeze Theorem.