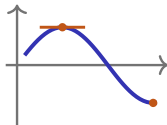


Analytical Applications of Differentiation

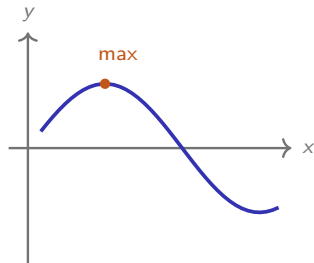
AP Calculus AB ■ Unit 5

AP Calculus AB



What we will cover

- 1 The Mean Value Theorem
- 2 Critical points & extrema
- 3 The first derivative test
- 4 Concavity & inflection
- 5 Connecting f , f' and f''
- 6 Optimization



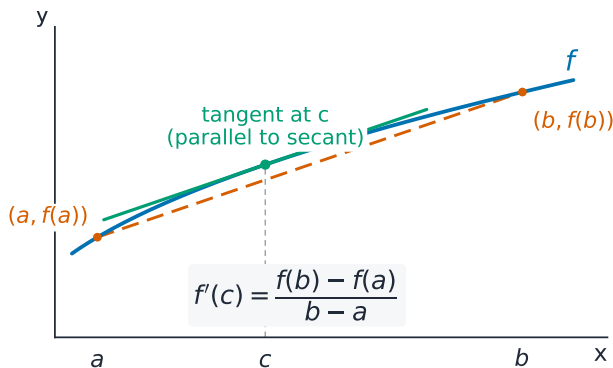


The Mean Value Theorem

If f is **continuous** on $[a, b]$ and **differentiable** on (a, b) , then for some c in (a, b) :

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Somewhere inside, the **instantaneous** rate equals the **average** rate – some tangent is **parallel to the secant**.



Exam skill – justifying the MVT

The **Mean Value Theorem** is an **existence** theorem, so the marks are in the justification:

- 1 state f is continuous on $[a, b]$ and differentiable on (a, b)
- 2 compute $\frac{f(b) - f(a)}{b - a}$
- 3 conclude “by the MVT there is a c in (a, b) with $f'(c)$ equal to that value”

Both hypotheses must be named – dropping one loses the mark even with the right c .

Verify the MVT for $f(x) = x^2$ on $[1, 3]$

A polynomial, so continuous and differentiable everywhere. Average rate $\frac{9 - 1}{2} = 4$; solve $f'(c) = 2c = 4$ to get $c = 2$, which lies in $(1, 3)$.



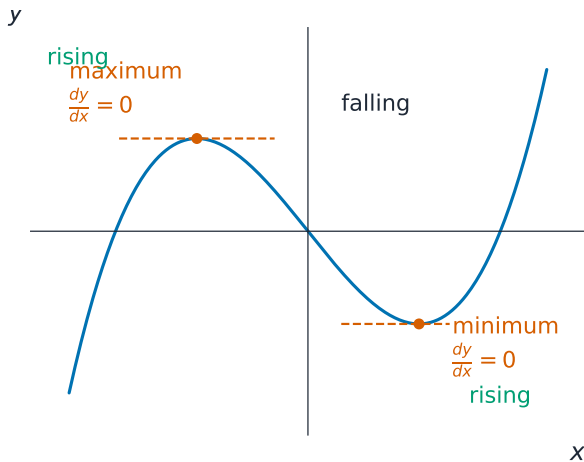
Extreme values and critical points

The **Extreme Value Theorem** :
a function **continuous** on a **closed**
interval attains **both** an absolute
max and an absolute min. A
critical point is an interior point
where $f'(x) = 0$ **or** f' does not
exist.



Garden fence

At a maximum or minimum the derivative is zero



At a maximum or minimum the derivative is zero

Increasing and decreasing intervals

The **first derivative** says where f rises or falls:

- $f'(x) > 0 \Rightarrow f$ is **increasing**
- $f'(x) < 0 \Rightarrow f$ is **decreasing**



Mountain extrema

The first derivative test

Classify a critical point c by how f' **changes sign** there:

f' turns $+$ $\rightarrow$ $-$
local maximum

f' turns $-$ $\rightarrow$ $+$
local minimum

f' does **not** change sign
neither

Always state the **sign change** as your justification.

Sketching a Function and Its Derivatives

Key features of f , f' , and f'' mirror each other.

- f increasing $\Leftrightarrow f'$ above the axis; f has a local max $\Leftrightarrow f'$ crosses from $+$ to $-$.
- f concave up $\Leftrightarrow f'$ increasing $\Leftrightarrow f''$ above the axis.

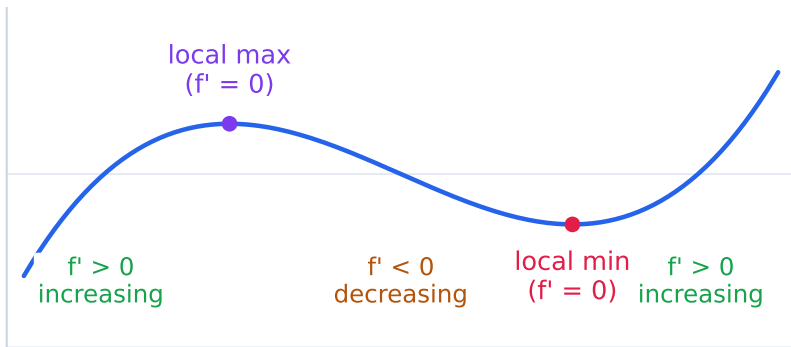
Worked example – classify the critical points

$$f(x) = x^3 - 3x^2$$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$, zero at $x = 0, 2$. Signs: $f' > 0$ for $x < 0$; $f' < 0$ on $(0, 2)$; $f' > 0$ for $x > 2$. $+$ $\rightarrow$ $-$ at $x = 0$: **local max**, $f(0) = 0$.

$- \rightarrow +$ at $x = 2$: **local min**, $f(2) = -4$.

Worked example – classify the critical points, in detail



$f' > 0 \Rightarrow$ increasing · $f' < 0 \Rightarrow$ decreasing · $f' = 0$ candidates for extrema

$$f(x) = x^3 - 3x^2 \quad f'(x) = 3x^2 - 6x = 3x(x - 2), \text{ zero at } x = 0, 2.$$

The candidates test – absolute extrema

On a **closed** interval, absolute extrema occur only at **critical points or endpoints**:

- 1 list every critical point in $[a, b]$ **and** both endpoints
- 2 evaluate f at each candidate
- 3 largest is the absolute max; smallest the absolute min

Show the **table of values** – the comparison *is* the argument.

endpoints a, b
+
every critical point
= the full candidate list

Extrema: local, global, and how to justify

Local (relative) extrema

Local (relative) extrema are the highest or lowest value in some neighbourhood – and by Fermat's theorem they occur only at critical points.

Absolute extrema

On a closed interval these are found by the candidates test: evaluate at every critical point *and* both endpoints.

Justifying

The first-derivative test names the sign change of f' ; the second-derivative test uses the sign of f'' at the critical point.

What earns the mark

State the value *and* the reason – “ f' changes from positive to negative at $x = 2$, so $f(2)$ is a local maximum”.

A critical point is a candidate, never a conclusion. The justification is the marked part of an AP answer.



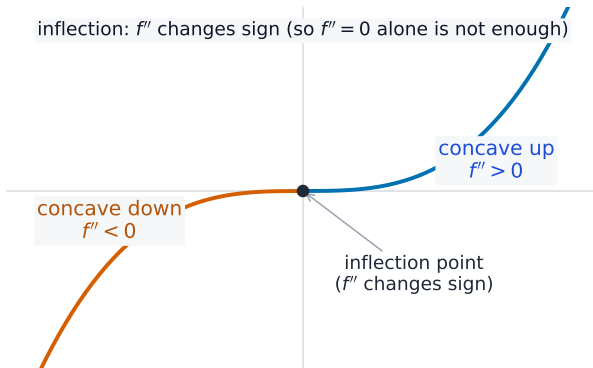
Concavity and points of inflection

The **second derivative** describes bending:

- $f'' > 0$: **concave up** (f' increasing)
- $f'' < 0$: **concave down** (f' decreasing)

A **point of inflection** is where f'' **changes sign** – not merely where $f'' = 0$.

Concavity and points of inflection, in detail



Concavity comes from the sign of the second derivative; it flips at an inflection point

The second derivative test

An alternative at a critical point c with $f'(c) = 0$:

$$f''(c) > 0 \Rightarrow \text{local minimum}$$

$$f''(c) < 0 \Rightarrow \text{local maximum}$$

$$f''(c) = 0 \Rightarrow \text{inconclusive}$$

If $f''(c) = 0$, fall back on the **first derivative test**.

Special case: if a continuous function has only **one** critical point on an interval and it is a local extremum, it is the **absolute** extremum there.

Connecting f , f' and f''

A very common setup: you are given the graph of f' and asked about f .

f **increasing** $\Leftrightarrow f'$ **above the axis**

f has a **local max** $\Leftrightarrow$
 f' crosses $+$ $\rightarrow$ $-$

f **concave up** $\Leftrightarrow f'$ **increasing**

f has an **inflection** $\Leftrightarrow$
 f' has a local extremum

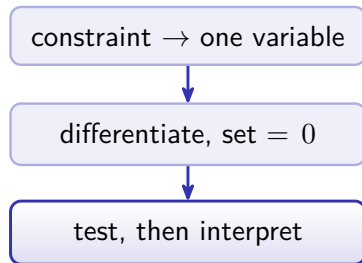
Answer questions about f using the **height** and **slope** of the f' graph.



Solving an optimization problem

Optimization is the same machinery aimed at a real goal:

- 1 write the quantity as a function of **one** variable (use the constraint to eliminate the rest)
- 2 state the allowed interval
- 3 find and **test** the critical points
- 4 answer with **units** and interpretation



Worked example – the farmer's fence

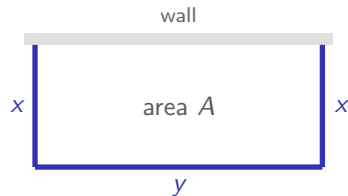
100 m of fence, three sides (a wall is the fourth). Maximize the area.

Ends x , side y : constraint $2x + y = 100$, so $y = 100 - 2x$.

$$A(x) = x(100 - 2x) = 100x - 2x^2$$

$$A'(x) = 100 - 4x = 0 \Rightarrow x = 25$$

$A'' = -4 < 0$, a maximum: $y = 50$ and $A = 1250 \text{ m}^2$.



Behaviour of implicit relations

It all extends to **implicitly** defined relations:

- horizontal tangent where $\frac{dy}{dx} = 0$
- vertical tangent where $\frac{dy}{dx}$ is **undefined**

$\frac{dy}{dx}$ usually involves **both** x and y – substitute it back in when finding $\frac{d^2y}{dx^2}$, then read concavity from the sign.



Unit 5 – key takeaways

1

MVT

some tangent is parallel to the secant – name both hypotheses

2

Critical points

$f' = 0$ or undefined: candidates only, so test each

3

The two tests

f' sign change, or $f'' > 0$ min / $f'' < 0$ max

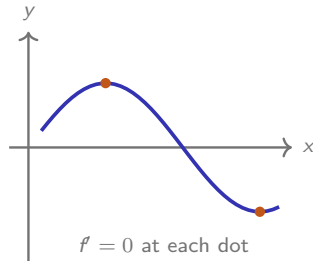
4

Justify

cite the sign of f' or f'' – the reasoning earns the marks

Check yourself

- 1 f' turns $-$ to $+$ at c . What is c ?
- 2 $f'(c) = 0$ at a critical point. What does the test tell you?
- 3 Where can an absolute max on $[a, b]$ occur?
- 4 Is every critical point an extremum?



Answers 1. a local minimum 2. inconclusive – use the f' test 3. a critical point or an endpoint 4. no