5.9 Connecting f, f-prime, and f-double-prime

Name: _____ Class: _____ Date: _____ **Total: 9 marks**

KEY WORDS local minimum • concave up • decreasing • increasing

Objective

Build the skills to answer exam questions on **connecting a function, its first derivative, and its second derivative**.

You must be able to:

- read f, f', and f'' together to describe a graph

1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later.

Connecting f, f', f''

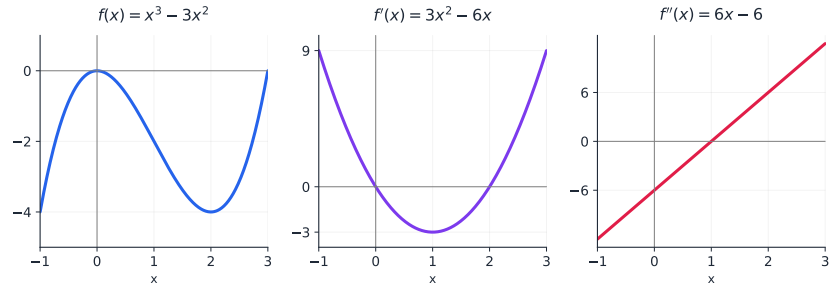
- f' tells increasing/decreasing;
- f'' tells concavity;
- $f' = 0$ and $f'' > 0$ are sufficient second-derivative-test conditions for a local **min**, not necessary conditions;
- an **inflection** point is a continuous point where concavity changes; inspect neighbouring intervals.

Example

$f(x) = x^3$: $f'(x) = 3x^2$ (zero at 0), $f''(x) = 6x$ (zero at 0).

Worked example: connect the same three graphs

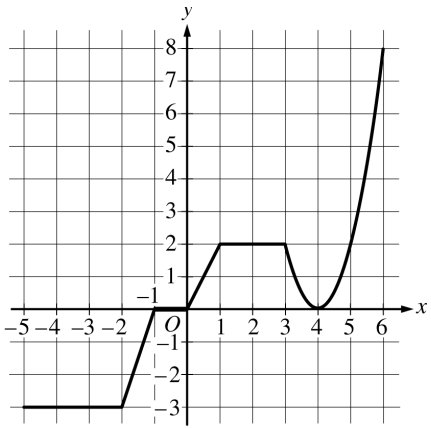
For $f(x) = x^3 - 3x^2$, the derivative is $f'(x) = 3x(x-2)$ and the second derivative is $f''(x) = 6x - 6$.



On $(-1, 0)$, $(0, 2)$ and $(2, 3)$, the signs of f' are positive, negative and positive. Hence f increases, decreases and increases. The sign changes give a local maximum at $(0, 0)$ and minimum at $(2, -4)$. On $(-1, 0)$ the function is increasing but concave down: $f' > 0$ while f' decreases. Concavity changes at 1 because f'' changes from negative to positive; the inflection is $(1, -2)$, where $f'(1) = -3$.

Adding 5 to f leaves both derivatives unchanged. The shape and turning-point x coordinates stay the same; the vertical position needs a given function value.

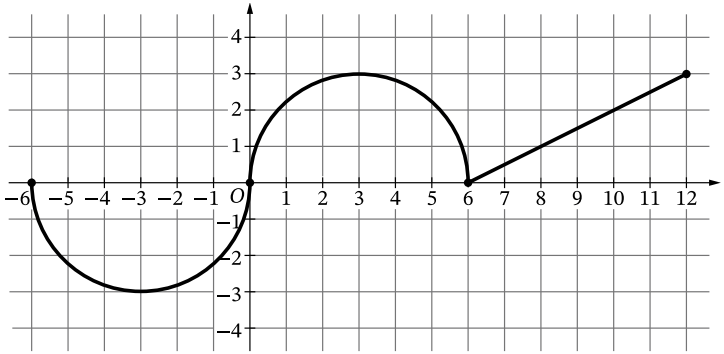
Original 2018 AB Q 3(c–d): read the displayed derivative



Graph of g

The given g is f'. For f to be increasing and concave up, g must be positive and increasing: $(0, 1)$ and $(4, 6)$. At 4, g changes from decreasing to increasing, so f has an inflection. Since g is positive on both sides of 4, f has no extremum there. The official scheme awards both the interval answer and its reason.

■ **Original 2025 AB Q 4B: the notation changes**



Graph of f

Here $g(x) = \int_6^x f(t)\,dt$, so $g' = f$. The inflections of g occur at $x = -3, 3, 6$: the displayed f changes decreasing/increasing behaviour there. At 0, f increases on both sides, so g remains concave up. At 6 the change of concavity is valid even though f' is undefined at its corner. The official scheme requires evidence tied to the given graph, not simply naming a change in concavity.

■ **Worked counterexample: a zero is only a candidate**

For $h(x) = x^4$, $h''(x) = 12x^2$ is positive on both sides of 0, so there is no inflection. Yet $h(x) \geq h(0) = 0$, so 0 is a minimum even though $h''(0) = 0$. For $k(x) = x^3$, $k''(x) = 6x$ changes from negative to positive, giving an inflection at 0. Check neighbouring concavity; the second derivative need not exist at the point itself. The function must be continuous at an inflection point.

2 Practice

2.1 State what f'' tells you about f. [1]

2.2 State sufficient conditions on f' and f'' for the second derivative test to prove a local minimum. [1]

2.3 For $f(x) = x^3$, state where $f' = 0$ and where $f'' = 0$. [2]

3 Exam-style questions

3.1 f'' gives information about a graph's [1]

☐ **A** slope
B concavity
C x-intercepts
D domain

3.2 Which conditions prove a local minimum of f by the second derivative test? [1]

☐ **A** $f' = 0,\ f'' > 0$
B $f' = 0,\ f'' < 0$
C $f'' = 0$ only
D $f = 0$

3.3 $f(x) = x^3$.

(a) Find $f'(x)$. [1]

(b) State where $f' = 0$. [1]

(c) State where $f'' = 0$. [1]