8.2 Motion: Position, Velocity, and Acceleration

Name: _____ Class: _____ Date: _____ **Total: 15 marks**

KEY WORDS total distance • displacement • net change

Objective

Build the skills to answer exam questions on **motion with integrals** – recovering velocity and position, and finding distance.

You must be able to:

- integrate acceleration to get velocity and velocity to get position
- find **displacement** $\int_a^b v\,dt$ and **total distance** $\int_a^b |v|\,dt$
- use an initial condition to fix the constant

1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later – follow the steps and you can do the Practice and Exam-style questions yourself.

■ Integrating the motion chain

Acceleration integrates to velocity; velocity integrates to position:

$$v(t) = v(t_0) + \int_{t_0}^{t} a(u)\,du, \qquad s(t) = s(t_0) + \int_{t_0}^{t} v(u)\,du.$$

These definite-integral forms include the initial values. In an indefinite-integral method, add a constant at each integration and fix it using the matching initial condition.

■ Displacement

Displacement over $[a, b]$ is the net change in position:

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

It can be negative (ended left of the start).

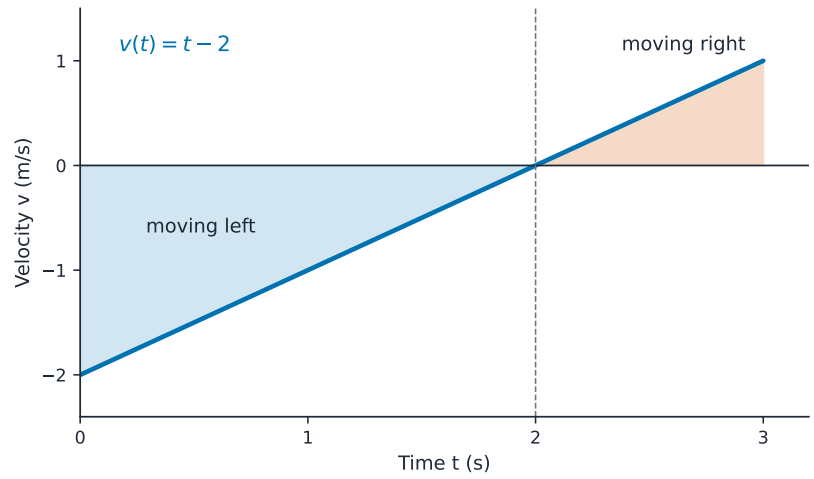
■ Total distance

Total distance counts all motion regardless of direction:

$$\int_a^b |v(t)|\,dt.$$

Split the integral where v changes sign, integrate each piece, and add the **absolute** amounts.

■ Read the sign before adding motion



Teacher-created model: velocity is negative before two seconds and positive afterwards; the particle changes direction.

For $v(t) = t - 2$ m/s on $[0, 3]$, solve $v(t) = 0$ and test either side. The particle moves left on $[0, 2)$ and right on $(2, 3]$.

Displacement is signed:

$$\Delta x = \int_0^3 v(t)\,dt = \left[\frac{t^2}{2} - 2t\right]_0^3 = -1.5 \text{ m}.$$

Total distance counts both journeys positively. Split at the sign change:

$$D = \int_0^3 |v(t)|\,dt = -\int_0^2 (t-2)\,dt + \int_2^3 (t-2)\,dt.$$

$$D = |-2| + \frac{1}{2} = 2.5 \text{ m}.$$

The particle travels two metres left and half a metre right. Taking only $|\int_0^3 v(t)\,dt| = 1.5$ m loses the return journey.

Position needs a starting position. If $x(0) = 5$ m, then

$$x(3) = x(0) + \int_0^3 v(t)\,dt = 5 - 1.5 = 3.5 \text{ m}.$$

The final position is right of the origin, even though displacement is negative.

A zero of velocity is a candidate for a turn. Check signs: $v(t) = (t-1)^2$ is positive on both sides of $t = 1$, so the particle stops momentarily without changing direction.

■ Recover velocity with an initial condition

For $a(t) = 4$ m/s^2 and $v(0) = 1$ m/s:

$$v(t) = v(0) + \int_0^t a(u)\,du = 1 + \int_0^t 4\,du = 1 + 4t.$$

Here t is in seconds and $v(t)$ is in metres per second. Equivalently, integrate to $4t + C$ and use $v(0) = 1$ to find $C = 1$. Each integration needs its own initial condition.

■ Original exam applications

2025 AB Q5D (calculator not allowed). The original gives $v_J(t) = 2t(t^2-1)^3$ and $x_J(0) = 7$, without physical units. Recover position, using $u = t^2 - 1$ and $du = 2t\,dt$:

$$x_J(2) = x_J(0) + \int_0^2 v_J(t)\,dt = 7 + \int_{-1}^3 u^3\,du.$$

$$x_J(2) = 7 + \left[\frac{u^4}{4}\right]_{-1}^3 = 7 + \frac{81-1}{4} = 27.$$

The change in position is 20, while final position is 27. Keep the initial position outside the integral.

2014 AB Q4(c). At times $2, 5, 8, 12$ min, train velocities are $100, 40, -120, -150$ m/min. Starting position is $s_A(2) = 300$ m east of the origin. Write the exact position expression, then estimate the signed change by the three specified trapezoids:

$$s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = 300 + \int_2^{12} v_A(t)\,dt.$$

$$\Delta s \approx 3\frac{100+40}{2} + 3\frac{40-120}{2} + 4\frac{-120-150}{2} = -450 \text{ m}.$$

$$s_A(12) \approx 300 - 450 = -150 \text{ m}.$$

This means about 150 m west of the origin. Preserve the unequal interval widths and signs. Sampled velocities do not locate every zero, so absolute trapezoids do not give exact total distance.

2 Practice

Now apply the methods above.

2.1 A particle has velocity $v(t) = 6t$. Find its displacement over $0 \le t \le 3$. [2]

2.2 With $a(t) = 4$ and $v(0) = 1$, find $v(t)$. [2]

2.3 A particle has $v(t) = 2t - 6$. State the time it changes direction. [1]

3 Exam-style questions

3.1 The total distance travelled over $[a, b]$ is [1]

☐ **A** $\int_a^b v\,dt$

B $\int_a^b |v|\,dt$

C $v(b) - v(a)$

D $\int_a^b a\,dt$

3.2 A particle moves with velocity $v(t) = 3t^2 - 12$ m/s.

(a) Find the displacement over $0 \le t \le 3$. [2]

(b) At what time does the particle change direction? [2]

3.3 A particle has velocity $v(t) = t - 3$ on $0 \leq t \leq 5$.

(a) Find the displacement. [2]

(b) Find the total distance travelled. [3]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\displaystyle\int_0^3 6t\,dt = \left[3t^2\right]_0^3 = 27$

2.2

- $v(t) = 4t + C$; $v(0) = C = 1$, so $v(t) = 4t + 1$

2.3

- $v = 0$ at $2t - 6 = 0$, i.e. $t = 3$

3.1 B

- total distance is $\int |v|\,dt$

3.2

(a)

- $\displaystyle\int_0^3 (3t^2 - 12)\,dt = \left[t^3 - 12t\right]_0^3 = 27 - 36 = -9 \text{ m}$

(b)

- $v = 0$: $3t^2 = 12 \Rightarrow t = 2$. Velocity is negative before 2 and positive afterwards on the given interval, so direction changes at $t = 2$ s.

3.3

(a)

- $\displaystyle\int_0^5 (t - 3)\,dt = \left[\tfrac{t^2}{2} - 3t\right]_0^5 = 12.5 - 15 = -2.5$

(b)

- $v < 0$ on $[0, 3)$, $v > 0$ on $(3, 5]$; distance $D = \left|\int_0^3 (t-3)\,dt\right| + \int_3^5 (t-3)\,dt = |-4.5| + 2 = 6.5$