

8.2 Motion: Position, Velocity, and Acceleration

Name: _____ Class: _____ Date: _____

Total: 15 marks

KEY WORDS total distance • displacement • net change

Objective

Build the skills to answer exam questions on **motion with integrals** – recovering velocity and position, and finding distance.

You must be able to:

- integrate acceleration to get velocity and velocity to get position
- find **displacement** $\int_a^b v \, dt$ and **total distance** $\int_a^b |v| \, dt$
- use an initial condition to fix the constant

1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later – follow the steps and you can do the Practice and Exam-style questions yourself.

■ Integrating the motion chain

Acceleration integrates to velocity; velocity integrates to position:

$$v(t) = v(t_0) + \int_{t_0}^t a(u) \, du, \quad s(t) = s(t_0) + \int_{t_0}^t v(u) \, du.$$

These definite-integral forms include the initial values. In an indefinite-integral method, add a constant at each integration and fix it using the matching initial condition.

■ Displacement

Displacement over $[a, b]$ is the net change in position:

$$\int_a^b v(t) \, dt = s(b) - s(a).$$

It can be negative (ended left of the start).

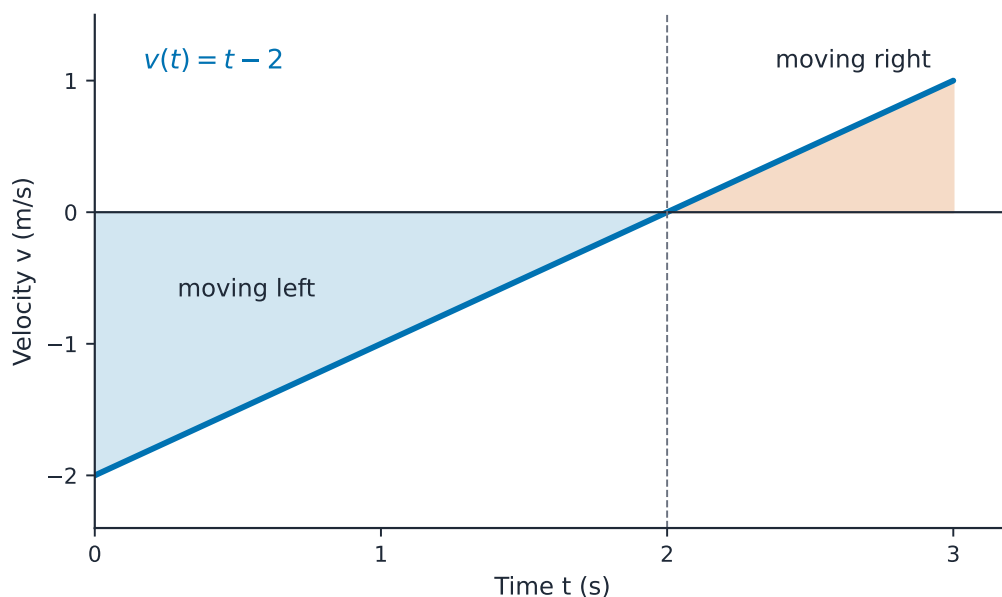
■ Total distance

Total distance counts all motion regardless of direction:

$$\int_a^b |v(t)| dt.$$

Split the integral where v changes sign, integrate each piece, and add the **absolute** amounts.

■ Read the sign before adding motion



Teacher-created model: velocity is negative before two seconds and positive afterwards; the particle changes direction.

For $v(t) = t - 2$ m/s on $[0, 3]$, solve $v(t) = 0$ and test either side. The particle moves left on $[0, 2)$ and right on $(2, 3]$.

Displacement is signed:

$$\Delta x = \int_0^3 v(t) dt = \left[\frac{t^2}{2} - 2t \right]_0^3 = -1.5 \text{ m.}$$

Total distance counts both journeys positively. Split at the sign change:

$$D = \int_0^3 |v(t)| dt = - \int_0^2 (t - 2) dt + \int_2^3 (t - 2) dt.$$

$$D = |-2| + \frac{1}{2} = 2.5 \text{ m.}$$

The particle travels two metres left and half a metre right. Taking only $|\int_0^3 v(t) dt| = 1.5$ m loses the return journey.

Position needs a starting position. If $x(0) = 5$ m, then

$$x(3) = x(0) + \int_0^3 v(t) dt = 5 - 1.5 = 3.5 \text{ m.}$$

The final position is right of the origin, even though displacement is negative.

A zero of velocity is a candidate for a turn. Check signs: $v(t) = (t - 1)^2$ is positive on both sides of $t = 1$, so the particle stops momentarily without changing direction.

■ **Recover velocity with an initial condition**

For $a(t) = 4 \text{ m/s}^2$ and $v(0) = 1 \text{ m/s}$:

$$v(t) = v(0) + \int_0^t a(u) du = 1 + \int_0^t 4 du = 1 + 4t.$$

Here t is in seconds and $v(t)$ is in metres per second. Equivalently, integrate to $4t + C$ and use $v(0) = 1$ to find $C = 1$. Each integration needs its own initial condition.

■ **Original exam applications**

2025 AB Q5D (calculator not allowed). The original gives $v_J(t) = 2t(t^2 - 1)^3$ and $x_J(0) = 7$, without physical units. Recover position, using $u = t^2 - 1$ and $du = 2t dt$:

$$x_J(2) = x_J(0) + \int_0^2 v_J(t) dt = 7 + \int_{-1}^3 u^3 du.$$

$$x_J(2) = 7 + \left[\frac{u^4}{4} \right]_{-1}^3 = 7 + \frac{81 - 1}{4} = 27.$$

The change in position is 20, while final position is 27. Keep the initial position outside the integral.

2014 AB Q4(c). At times 2, 5, 8, 12 min, train velocities are 100, 40, -120 , -150 m/min. Starting position is $s_A(2) = 300$ m east of the origin. Write the exact position expression, then estimate the signed change by the three specified trapezoids:

$$s_A(12) = s_A(2) + \int_2^{12} v_A(t) dt = 300 + \int_2^{12} v_A(t) dt.$$

$$\Delta s \approx 3 \frac{100 + 40}{2} + 3 \frac{40 - 120}{2} + 4 \frac{-120 - 150}{2} = -450 \text{ m.}$$

$$s_A(12) \approx 300 - 450 = -150 \text{ m.}$$

This means about 150 m west of the origin. Preserve the unequal interval widths and signs. Sampled velocities do not locate every zero, so absolute trapezoids do not give exact total distance.

2 Practice

Now apply the methods above.

2.1 A particle has velocity $v(t) = 6t$. Find its displacement over $0 \leq t \leq 3$. [2]

2.2 With $a(t) = 4$ and $v(0) = 1$, find $v(t)$. [2]

2.3 A particle has $v(t) = 2t - 6$. State the time it changes direction. [1]

3 Exam-style questions

3.1 The total distance travelled over $[a, b]$ is [1]

- ☐ **A** $\int_a^b v \, dt$
- B** $\int_a^b |v| \, dt$
- C** $v(b) - v(a)$
- D** $\int_a^b a \, dt$

3.2 A particle moves with velocity $v(t) = 3t^2 - 12$ m/s.

(a) Find the displacement over $0 \leq t \leq 3$. [2]

(b) At what time does the particle change direction? [2]

3.3 A particle has velocity $v(t) = t - 3$ on $0 \leq t \leq 5$.

(a) Find the displacement. [2]

(b) Find the total distance travelled. [3]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\int_0^3 6t \, dt = [3t^2]_0^3 = 27$

2.2

- $v(t) = 4t + C$; $v(0) = C = 1$, so $v(t) = 4t + 1$

2.3

- $v = 0$ at $2t - 6 = 0$, i.e. $t = 3$

3.1 B

- total distance is $\int |v| \, dt$

3.2

(a)

- $\int_0^3 (3t^2 - 12) \, dt = [t^3 - 12t]_0^3 = 27 - 36 = -9 \text{ m}$

(b)

- $v = 0$: $3t^2 = 12 \Rightarrow t = 2$. Velocity is negative before 2 and positive afterwards on the given interval, so direction changes at $t = 2$ s.

3.3

(a)

- $\int_0^5 (t - 3) \, dt = [\frac{t^2}{2} - 3t]_0^5 = 12.5 - 15 = -2.5$

(b)

- $v < 0$ on $[0, 3)$, $v > 0$ on $(3, 5]$; distance $D = |\int_0^3 (t - 3) \, dt| + \int_3^5 (t - 3) \, dt = |-4.5| + 2 = 6.5$