6.10 Long Division and Completing the Square

Name: _____ Class: _____ Date: _____ **Total: 14 marks**

KEY WORDS complete the square • completing the square • polynomial long division

Objective

Build the skills to answer exam questions on **integrating with long division and completing the square** – rewriting a rational integrand into an integrable form.

You must be able to:

- use **polynomial long division** when the numerator's degree is $\geq$ the denominator's
- **complete the square** in a denominator to reach an **arctangent** form
- recognise $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later – follow the steps and you can do the Practice and Exam-style questions yourself.

■ Long division first

$\displaystyle\int \frac{x^2}{x+1}\,dx$ is top-heavy. Divide: $\displaystyle\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$. Then

$$\int \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{x^2}{2} - x + \ln|x+1| + C.$$

■ A simple 1/x-type

$\displaystyle\int \frac{1}{x-3}\,dx = \ln|x-3| + C$ (substitution $u = x-3$). Watch for these after dividing.

■ Completing the square

$\displaystyle\int \frac{1}{x^2+4x+5}\,dx$: complete the square, $x^2+4x+5 = (x+2)^2+1$. With $u = x+2$,

$$\int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+2) + C.$$

■ The arctangent template

$\displaystyle\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$. For $\displaystyle\int \frac{1}{x^2+9}\,dx = \tfrac{1}{3}\arctan\tfrac{x}{3} + C$.

2 Practice

Now apply the methods above.

2.1 Use long division to rewrite $\dfrac{x+3}{x+1}$ as $1 + \dfrac{?}{x+1}$. [2]

2.2 Find $\displaystyle\int \frac{1}{x^2+16}\,dx$. [2]

2.3 Complete the square: write $x^2+6x+13$ in the form $(x+a)^2+b$. [2]

3 Exam-style questions

3.1 $\displaystyle\int \frac{1}{x^2+25}\,dx =$ [1]

☐ **A** $\arctan(x) + C$
B $\frac{1}{5}\arctan\frac{x}{5} + C$
C $\ln(x^2+25) + C$
D $5\arctan(5x) + C$

3.2 Find $\int \frac{x+2}{x+1}\,dx$. [3]

3.3 Find $\int \frac{1}{x^2+2x+10}\,dx$ by completing the square. [4]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\dfrac{x+3}{x+1} = 1 + \dfrac{2}{x+1}$

2.2

- $\frac{1}{4}\arctan\frac{x}{4} + C$

2.3

- $(x+3)^2 + 4$

3.1 B

- with $a = 5$, $\int \frac{dx}{x^2+25} = \frac{1}{5}\arctan\frac{x}{5} + C$

3.2

- $\dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$; $\int \left(1 + \frac{1}{x+1}\right)dx = x + \ln|x+1| + C$

3.3

- $x^2 + 2x + 10 = (x+1)^2 + 9$; with $u = x+1$, $\int \frac{du}{u^2+9} = \frac{1}{3}\arctan\frac{u}{3} = \frac{1}{3}\arctan\dfrac{x+1}{3} + C$