1.1 Introducing Calculus: Can Change Occur at an Instant?

Name: _____ Class: _____ Date: _____ **Total: 18 marks**

0 Vocabulary

Word	Example
± **change** /tʃeɪndʒ/	Calculus studies **change**, such as how fast a car speeds up.
accumulation /əˌkjuːmjʊˈleɪʃn/	Integration measures **accumulation**: how much builds up over time.
at a single instant /ætə ˈsɪŋgl ˈɪnstənt/	A derivative gives the rate of change **at a single instant**.
average rate of change /ˈævrɪdʒ reɪt ɒv tʃeɪndʒ/	The **average rate of change** is the change in y divided by the change in x. $\frac{\text{y}}{\text{x}}$
[ ] **interval** /ˈɪntəvl/	The **interval** from 2 to 5 includes every number between them. 2 5
instantaneous rate of change /ˌɪnstənˈteɪnɪəs reɪt ɒv tʃeɪndʒ/	The **instantaneous rate of change** is the rate at one exact moment.

exercises.lol exercises.com.cn

1 Key knowledge and worked examples

■ Average and instantaneous rates

idea	rule
average rate of change on $[a, b]$	$\dfrac{f(b)-f(a)}{b-a}$: the slope of the secant line
units	output units per input unit, e.g. meters per second
average rate on $[a, a+h]$	$\dfrac{f(a+h)-f(a)}{h}$
instantaneous rate at a	the limit of the average rates as $h \to 0$: the slope of the tangent line
order of work	expand, simplify, cancel h (allowed because $h \neq 0$), then let $h \to 0$

Worked example 1: an average rate. Find the average rate of change of $f(x) = x^3$ on $[1, 3]$.

1. Known: $f(1) = 1$, $f(3) = 27$.
2. Rule: change in output over change in input.
3. Numbers: $\dfrac{27-1}{3-1}$.
4. Result: 13.

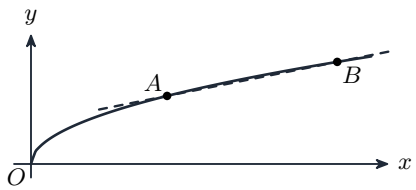
Worked example 2: an instant. A particle has position $s(t) = t^2 + t$ meters. Find its velocity at $t = 3$.

1. Known: $s(3) = 12$, $s(3+h) = 12 + 7h + h^2$.
2. Rule: average velocity on $[3, 3+h]$, then the limit.
3. Numbers: $\dfrac{7h+h^2}{h} = 7 + h$ for $h \neq 0$; as $h \to 0$, $7 + h \to 7$.
4. Result: 7 **m/s**.

Answer shapes: always give units with a rate; a negative rate means the quantity is decreasing.

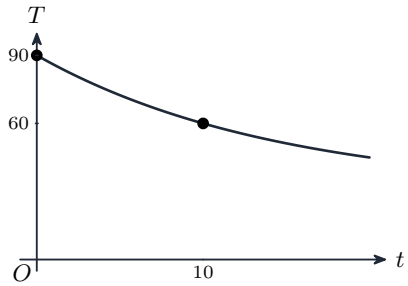
2 Practice

2.1 Find the average rate of change of $g(x) = \sqrt{x}$ on the interval $[4, 9]$. [2]



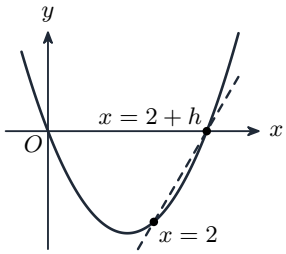
exercises.lol exercises.com.cn

2.2 A cup of tea is cooling. Its temperature is 90°C at $t = 0$ and 60°C at $t = 10$, where t is in minutes. Find the average rate of change of the temperature over these 10 minutes. Give units. [2]



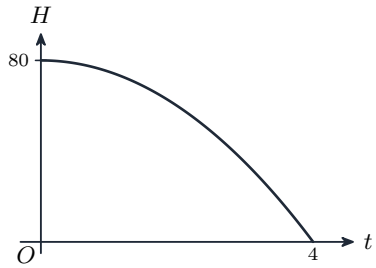
3 Exam-style questions

3.1 Let $f(x) = x^2 - 3x$. Show that the average rate of change of f on $[2, 2 + h]$ is $1 + h$ for $h \neq 0$. Hence state the instantaneous rate of change of f at $x = 2$. [3]



3.2 A stone is dropped from a cliff. Its height above the sea after t seconds is $H(t) = 80 - 5t^2$ meters, for $0 \leq t \leq 4$.

(a) Find the average velocity of the stone over $1 \leq t \leq 3$. Give units. [2]



(b) Show that the average velocity over $[2, 2 + k]$ is $-20 - 5k$ for $k \neq 0$. [2]

exercises.lol exercises.com.cn

(c) Find the velocity of the stone at $t = 2$. Explain why you cannot put $k = 0$ into $\dfrac{H(2+k) - H(2)}{k}$ before you simplify it. [3]

(d) Find the time at which the velocity of the stone equals its average velocity over the whole fall, $0 \leq t \leq 4$. [4]

exercises.lol exercises.com.cn

Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

- $\dfrac{g(9)-g(4)}{9-4}=\dfrac{3-2}{5}$
- $\mathbf{\frac{1}{5}}$

2.2

- $\dfrac{60-90}{10-0}$
- **-3 degrees Celsius per minute**

3.1

- $f(2+h)=4+4h+h^2-6-3h=-2+h+h^2$ and $f(2)=-2$
- $\dfrac{h+h^2}{h}=1+h$
- as $h\to 0$: instantaneous rate **1**

3.2

(a) $H(1)=75$, $H(3)=35$; $\dfrac{35-75}{2}=\mathbf{-20\ m/s}$

(b) $H(2+k)=80-5(4+4k+k^2)=60-20k-5k^2$ and $H(2)=60$; $\dfrac{-20k-5k^2}{k}=-20-5k$

(c) as $k\to 0$, $-20-5k\to -20$; velocity $\mathbf{-20\ m/s}$; at $k=0$ the quotient is $\frac{0}{0}$, which has no value

(d) average over $[0,4]$: $\dfrac{0-80}{4}=-20$ m/s; $\dfrac{H(t+k)-H(t)}{k}=\dfrac{-10tk-5k^2}{k}=-10t-5k$; velocity $-10t$; $-10t=-20$ gives $\mathbf{t=2\ s}$

exercises.lol exercises.com.cn