

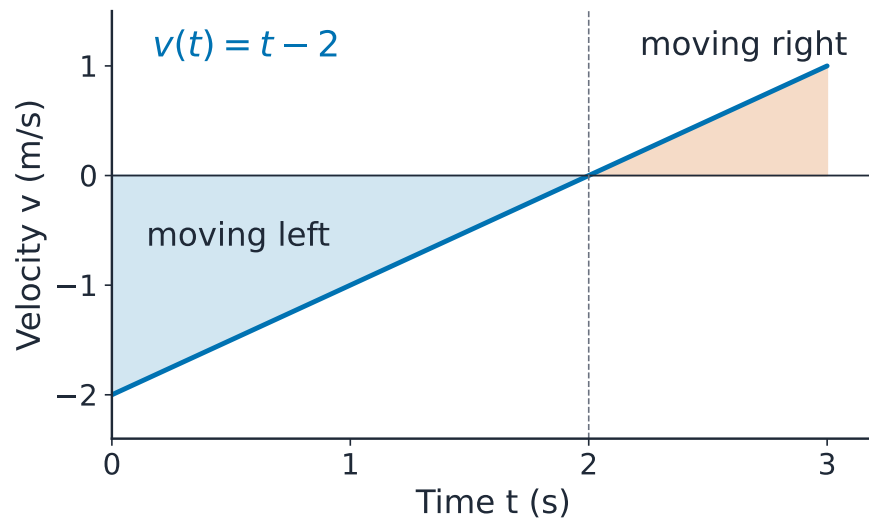
Exercise walk-through

8.2

Connecting Position, Velocity, and Acceleration of Functions Using Integrals

AP Calculus AB

Read velocity before integrating



Your turn: teacher-created model

For $0 \leq t \leq 3$, find when direction changes. Predict whether displacement and total distance are equal. Explain using the graph.

Direction comes from the sign of velocity

$$v(t) = t - 2 = 0 \quad \Rightarrow \quad t = 2 \text{ s.}$$

Interval	$[0, 2)$	$(2, 3]$
Sign of v	$-$	$+$
Direction	left	right

Choose the quantity

Displacement counts signed change in position.

Total distance counts both journeys positively.

Split at $t = 2$: this zero is accompanied by a sign change.

Evaluate the two journeys separately

$$\Delta x = \int_0^3 v(t) dt = \left[\frac{t^2}{2} - 2t \right]_0^3 = -1.5 \text{ m.}$$

$$D = \int_0^3 |v(t)| dt = - \int_0^2 (t - 2) dt + \int_2^3 (t - 2) dt.$$

$$D = |-2| + \frac{1}{2} = 2.5 \text{ m.}$$

Interpret the result

The particle travels 2 m left, then 0.5 m right. It ends 1.5 m left of its start, after travelling 2.5 m.

$\left| \int v dt \right|$ gives the magnitude of displacement; it misses the return journey.

Position needs the starting position

Your turn

The same particle starts at $x(0) = 5$ m. Find $x(3)$. Is $x(3)$ its displacement?

$$x(b) = x(a) + \int_a^b v(t) dt.$$

The integral changes the starting position; it does not replace it.

Add signed change to the initial position

Recover position

$$x(3) = x(0) + \int_0^3 v(t) dt.$$

$$x(3) = 5 + (-1.5) = 3.5 \text{ m.}$$

Quantity	Value	Meaning
Final position	3.5 m	right of the origin
Displacement	-1.5 m	left of the starting point
Distance	2.5 m	length of the whole journey

Recover velocity from acceleration

Known quantities

$a(t) = 4 \text{ m/s}^2$ and $v(0) = 1 \text{ m/s}$. Find $v(t)$.

$$v(t) = v(0) + \int_0^t a(u) \, du.$$

$$v(t) = 1 + \int_0^t 4 \, du = 1 + 4t \quad \text{in metres per second.}$$

Equivalently, $v(t) = 4t + C$ and $v(0) = 1$ fixes $C = 1$. Each integration needs its own initial condition.

Does every zero mean a change of direction?

Test a claim

For $v(t) = (t - 1)^2$, on $0 \leq t \leq 2$, a student says: “The particle turns at $t = 1$ because $v(1) = 0$.” Decide whether the reason is sufficient.

A stop can occur without a turn

Check both sides

$v(t) > 0$ on both $(0, 1)$ and $(1, 2)$. The particle is momentarily at rest, but continues in the same direction.

Find zeros, then test the sign on each interval. Do not assume every root is a turning point.

Original exam: recover particle J position

2025 AB Q5D: calculator not allowed

Particle J has $v_J(t) = 2t(t^2 - 1)^3$ and $x_J(0) = 7$. Find $x_J(2)$, showing your work. The original gives no physical units.

- 1 Write an initial-position-plus-integral equation.
- 2 Identify a substitution from the derivative of $t^2 - 1$.

Original exam: set up before evaluating

$$x_J(2) = x_J(0) + \int_0^2 v_J(t) dt.$$

$$x_J(2) = 7 + \int_0^2 2t(t^2 - 1)^3 dt.$$

Substitution

$u = t^2 - 1$, so $du = 2t dt$.

$t = 0 \Rightarrow u = -1$; $t = 2 \Rightarrow u = 3$.

Keep 7 outside the integral. Velocity integrates to signed change.

Original exam: evaluate and interpret

$$x_J(2) = 7 + \int_{-1}^3 u^3 du.$$

$$x_J(2) = 7 + \left[\frac{u^4}{4} \right]_{-1}^3.$$

$$x_J(2) = 7 + \frac{81 - 1}{4} = 27.$$

The change in position is 20; the final position is 27. The official three points distinguish integrand, antiderivative and answer.

Original exam: position from a velocity table

2014 AB Q4(c)

Train A is 300 m east of the origin at $t = 2$ min. Use the three table intervals to estimate its position at $t = 12$ min by trapezoids.

t (min)	2	5	8	12
$v_A(t)$ (m/min)	100	40	-120	-150

Write the exact integral expression first. Preserve the signs and unequal interval widths.

Original exam: add the trapezoidal net change

$$s_A(12) = s_A(2) + \int_2^{12} v_A(t) dt = 300 + \int_2^{12} v_A(t) dt.$$

$$\Delta s \approx 3 \frac{100 + 40}{2} + 3 \frac{40 - 120}{2} + 4 \frac{-120 - 150}{2}.$$

$$\Delta s \approx 210 - 120 - 540 = -450 \text{ m.}$$

$$s_A(12) \approx 300 - 450 = -150 \text{ m.}$$

About 150 m west of the origin. Table samples alone do not locate every zero, so taking absolute trapezoids is not an exact distance calculation.

Exercises 2.1·2.3

2.1 A particle has velocity $v(t) = 6t$. Find its displacement over $0 \leq t \leq 3$. [2]

2.2 With $a(t) = 4$ and $v(0) = 1$, find $v(t)$. [2]

2.3 A particle has $v(t) = 2t - 6$. State the time it changes direction. [1]

Exercises 2.1·2.3

2.1 A particle has velocity $v(t) = 6t$. Find its displacement over $0 \leq t \leq 3$. [2]

2.1

- $\int_0^3 6t \, dt = [3t^2]_0^3 = 27$

2.2 With $a(t) = 4$ and $v(0) = 1$, find $v(t)$. [2]

2.2

- $v(t) = 4t + C$; $v(0) = C = 1$, so $v(t) = 4t + 1$

2.3 A particle has $v(t) = 2t - 6$. State the time it changes direction. [1]

2.3

- $v = 0$ at $2t - 6 = 0$, i.e. $t = 3$

Exercise 3.1

3.1 The total distance travelled over $[a, b]$ is

[1]

☐

A $\int_a^b v \, dt$

B $\int_a^b |v| \, dt$

C $v(b) - v(a)$

D $\int_a^b a \, dt$

Exercise 3.1

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[1]

☐

A $\int_a^b v \, dt$

B $\int_a^b |v| \, dt$

C $v(b) - v(a)$

D $\int_a^b a \, dt$

3.1 B

- total distance is $\int |v| \, dt$

Exercise 3.2 (a)

3.2 A particle moves with velocity $v(t) = 3t^2 - 12$ m/s.

(a) Find the displacement over $0 \leq t \leq 3$.

[2]

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3.2 A particle moves with velocity $v(t) = 3t^2 - 12$ m/s.

(a) Find the displacement over $0 \leq t \leq 3$.

[2]

(a)

- $$\int_0^3 (3t^2 - 12) dt = [t^3 - 12t]_0^3 = 27 - 36 = -9 \text{ m}$$

Exercise 3.2 (b)

3.2 A particle moves with velocity $v(t) = 3t^2 - 12$ m/s.

(b) At what time does the particle change direction?

[2]

Exercise 3.2 (b)

3.2 A particle moves with velocity $v(t) = 3t^2 - 12$ m/s.

(b) At what time does the particle change direction?

[2]

(b)

- $v = 0$: $3t^2 = 12 \Rightarrow t = 2$. Velocity is negative before 2 and positive afterwards on the given interval, so direction changes at $t = 2$ s.

Exercise 3.3 (a)

3.3 A particle has velocity $v(t) = t - 3$ on $0 \leq t \leq 5$.

(a) Find the displacement.

[2]

Exercise 3.3 (a)

3.3 A particle has velocity $v(t) = t - 3$ on $0 \leq t \leq 5$.

(a) Find the displacement.

[2]

(a)

- $$\int_0^5 (t - 3) dt = \left[\frac{t^2}{2} - 3t \right]_0^5 = 12.5 - 15 = -2.5$$

Exercise 3.3 (b)

3.3 A particle has velocity $v(t) = t - 3$ on $0 \leq t \leq 5$.

(b) Find the total distance travelled.

[3]

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3.3 A particle has velocity $v(t) = t - 3$ on $0 \leq t \leq 5$.

(b) Find the total distance travelled.

[3]

(b)

- $v < 0$ on $[0, 3)$, $v > 0$ on $(3, 5]$; distance $D = \left| \int_0^3 (t-3) dt \right| + \int_3^5 (t-3) dt = |-4.5| + 2 = 6.5$