Exercise walk-through

7.6
Finding General Solutions Using Separation of Variables

AP Calculus AB

Separation of variables

If $\frac{dy}{dx}$ factors into a function of x times a function of y, split them and integrate:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

> Add the constant of integration once – on the x-side.

all y one side,
all x the other

↓

integrate both sides

↓

the **general solution**

Worked example – separate, integrate, then pin it down

Solve $\frac{dy}{dx} = xy$ with $y(0) = 2$

$$\int \frac{dy}{y} = \int x\,dx \;\Rightarrow\; \ln|y| = \frac{x^2}{2} + C$$

$$\Rightarrow\; y = Ae^{x^2/2}$$

Applying $y(0) = 2$ gives $A = 2$:

$$y = 2e^{x^2/2}.$$

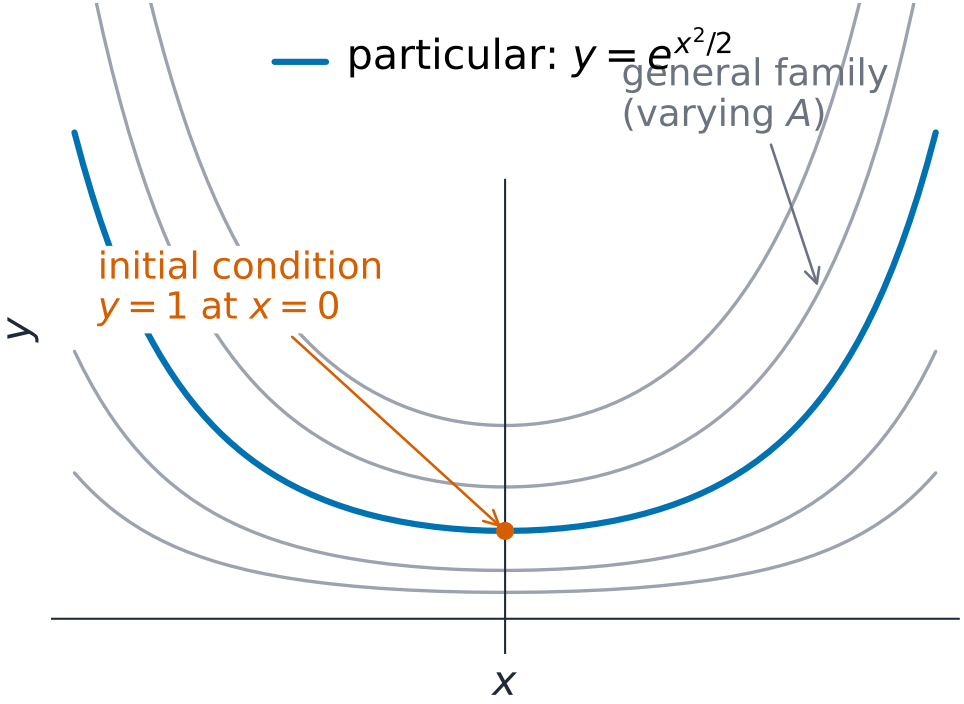
An **initial condition** (x_0, y_0) picks one curve from the family – there is exactly one solution through a given point.

From the family to the particular solution

The constant gives a family of curves; the initial condition selects the **particular solution** .

> Watch **domain restrictions** : keep the branch containing the initial point (e.g. the right sign of a square root).

From the family to the particular solution, in detail



The constant gives a family of curves; the initial condition selects the **particular solution** .

Exercises 2.1·2.3

2.1 Separate the variables in $\dfrac{dy}{dx} = 2xy$. [1]

2.2 Solve $\dfrac{dy}{dx} = \dfrac{1}{y}$ (find the general solution). [3]

2.3 Solve $\dfrac{dy}{dx} = 3y$ (general solution in terms of a constant A). [3]

Exercises 2.1·2.3

2.1 Separate the variables in $\dfrac{dy}{dx} = 2xy$. [1]

2.1

- $\dfrac{1}{y}\,dy = 2x\,dx$

2.2 Solve $\dfrac{dy}{dx} = \dfrac{1}{y}$ (find the general solution). [3]

2.2

- $y\,dy = dx$; $\frac{1}{2}y^2 = x + C$; $y^2 = 2x + C_1$

2.3 Solve $\dfrac{dy}{dx} = 3y$ (general solution in terms of a constant A). [3]

2.3

- $\dfrac{1}{y}\,dy = 3\,dx$; $\ln|y| = 3x + C$; $y = Ae^{3x}$

Exercise 3.1

3.1 After separating $\dfrac{dy}{dx} = ky$, integrating the left side gives [1]

A y

B $\ln|y|$

C $\dfrac{1}{y^2}$

D e^y

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3.1 After separating $\dfrac{dy}{dx} = ky$, integrating the left side gives [1]

A y

B $\ln|y|$

C $\dfrac{1}{y^2}$

D e^y

3.1 B

- $\int \frac{1}{y}\,dy = \ln|y|$

Exercise 3.2 (a)

3.2 Solve $\dfrac{dy}{dx} = xy^2$.

(a) Separate and integrate both sides. [3]

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(a)

- $\dfrac{1}{y^2}\,dy = x\,dx;\ -\dfrac{1}{y} = \dfrac{x^2}{2} + C$

Exercise 3.2 (b)

3.2 Solve $\dfrac{dy}{dx} = xy^2$.

(b) Solve for y in terms of x and a constant. [2]

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(b) Solve for y in terms of x and a constant. [2]

(b)

- $y = -\dfrac{1}{\frac{x^2}{2} + C} = \dfrac{-2}{x^2 + C_1}$

Exercise 3.3

3.3 Solve $\dfrac{dy}{dx} = \dfrac{\cos x}{y}$, giving the general solution. [3]

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3.3

- $y\,dy = \cos x\,dx$; $\frac{1}{2}y^2 = \sin x + C$; $y^2 = 2\sin x + C_1$