Exercise walk-through

1.2 Defining Limits and Using Limit Notation

AP Calculus AB

The limit theorems

If both limits exist, the limit of a combination is that combination of the limits:

$$\lim(f \pm g) = \lim f \pm \lim g$$

$$\lim(fg) = \lim f \cdot \lim g$$

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad (\text{bottom} \neq 0)$$

$$\lim g(f(x)) = g(\lim f(x)) \quad (g \text{ continuous})$$

The practical rule: try **direct substitution** first. A real number out means you are done.

The vocabulary a limit question uses

Piecewise-defined function	A **piecewise-defined function** uses different rules on different intervals – so the one-sided limits at a join must be checked separately.
Composite	For a **composite** $f(g(x))$, the limit passes inside when f is continuous there: $\lim f(g(x)) = f(\lim g(x))$.
Radical	A **radical** expression with an indeterminate form usually needs the conjugate – multiply top and bottom by it and the difference of squares clears the root.
Continuous on an interval	A function is **continuous on an interval** when it is continuous at every point of it – and one-sided at the endpoints.
Existence theorem	An **existence theorem** – the Intermediate Value Theorem, the Extreme Value Theorem – guarantees that something exists without saying where.

> The **instantaneous rate of change** is the limit of an average rate as the interval shrinks. That sentence is the whole idea of the derivative.

When substitution gives $\frac{0}{0}$

$\frac{0}{0}$ is an **indeterminate form** – it does not mean the limit fails. Rewrite into an **equivalent form**, then substitute.

> The cancelled factor is why the graph had a hole: the two functions agree everywhere except at c.

Three standard moves

Factor and cancel:

$$\lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$$

Multiply by the **conjugate**:

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \frac{1}{2}$$

Or use a trig identity.

Choosing a procedure

1. substitute – a real answer means done

2. $\frac{0}{0}$? rewrite, then substitute

3. $\frac{5}{0}$? infinite or DNE – check each side

4. $x \to \pm\infty$? compare the **dominant** terms

5. trapped between two functions? squeeze theorem

Choosing a method for a limit

1. substitute	if it gives a number, that is the limit – for a **continuous** function
2. factor and cancel	for $0/0$ caused by a common factor
3. rationalise	multiply by the conjugate when a surd causes the $0/0$
4. use identities	alternate forms of trigonometric functions often simplify what algebra cannot
5. squeeze	trap the function between two with the same limit
6. compare degrees	at infinity, the leading terms decide the **end behavior**

> This is a skill topic, not new content. And the same limit lives in a graph, a table and an algebraic form – move between all three.

Exercises 2.1·2.3

2.1 State what $\lim_{x \to a} f(x) = L$ means. [1]

2.2 State whether the limit must equal $f(a)$. [1]

2.3 For $f(x) = 2x$, state $\lim_{x \to 3} f(x)$. [2]

Exercises 2.1·2.3

2.1 State what $\lim_{x \to a} f(x) = L$ means. [1]

2.1

- $f(x)$ gets arbitrarily close to L as x approaches a

2.2 State whether the limit must equal $f(a)$. [1]

2.2

- no – the limit may differ from $f(a)$

2.3 For $f(x) = 2x$, state $\lim_{x \to 3} f(x)$. [2]

2.3

- $\lim_{x \to 3} 2x = 6$

Exercises 3.1·3.2

3.1 $\lim_{x \to a} f(x) = L$ means $f(x)$ approaches [1]

- [ ]
  A a
  B L as x approaches a
  C infinity
  D $f(a)$ always

3.2 The value of a limit [1]

- [ ]
  A always equals $f(a)$
  B may differ from $f(a)$
  C is always 0
  D never exists

Exercises 3.1·3.2

3.1 $\lim_{x \to a} f(x) = L$ means $f(x)$ approaches [1]

- **A** a
- **B** L as x approaches a
- **C** infinity
- **D** $f(a)$ always

3.1 B

- L as x approaches a.

3.2 The value of a limit [1]

- **A** always equals $f(a)$
- **B** may differ from $f(a)$
- **C** is always 0
- **D** never exists

3.2 B

- may differ from $f(a)$.

Exercise 3.3 (a)

3.3 $f(x) = x + 1$.

(a) State $\lim_{x \to 2} f(x)$. [1]

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(a)

- 3

Exercise 3.3 (b)

3.3 $f(x) = x + 1$.

(b) State $f(2)$. [1]

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3.3 $f(x) = x + 1$.

(b) State $f(2)$. [1]

(b)

- 3

Exercise 3.3 (c)

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(c) State whether they agree. [1]

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(c) State whether they agree. [1]

(c)

- yes

Exercise 3.4

3.4 Let $g(x) = x + 3$ for $x \neq 1$, and $g(1) = 8$. (a) Find the left- and right-hand limits at 1. [2]

(b) State the two-sided limit and explain whether g is continuous at 1. [2]

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3.4 Let $g(x) = x + 3$ for $x \neq 1$, and $g(1) = 8$. (a) Find the left- and right-hand limits at 1. [2]

(b) State the two-sided limit and explain whether g is continuous at 1. [2]

3.4

(a) Both one-sided limits are 4.

(b) The two-sided limit is 4; $g(1) = 8 \neq 4$, so continuity fails.