Plane transformations, coordinate reasoning and proof

Original skill preparation; versioned source scope

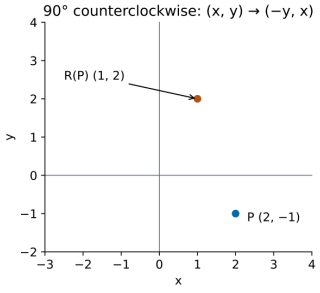
Plane Transformations, Coordinate Reasoning and Proof

- Apply each of the **rigid motions** 刚体变换: **translation** 平移, reflection and rotation.
- Use a **dilation** 位似 and **similarity** 相似, and decide on **congruence** 全等.
- Use distance, midpoint and slope, including the **perpendicular bisector** 垂直平分线.
- Give a reason that holds for every case, not one measured diagram.

Rigid motions

- **Translation** by (a, b): $(x, y) \to (x + a, y + b)$.
- Reflection in the x-axis: $(x, -y)$; in the y-axis: $(-x, y)$; in $y = x$: (y, x).
- Rotation about the origin by 90° anticlockwise: $(-y, x)$; by 180°: $(-x, -y)$.
- **Rigid motions** keep lengths and angles. The order of two motions can change the result.

Figure: Rigid motions



- A 90° counterclockwise rotation sends (x, y) to (−y, x)
- Match each part of the diagram to the text.

Dilations, similarity and congruence

- A **dilation** with scale factor k about the origin: (kx, ky). Lengths $\times k$, areas $\times k^2$, angles unchanged.
- Figures are congruent if a rigid motion maps one onto the other; similar if a rigid motion and a dilation do.
- Triangles with equal angles are similar; SSS, SAS, ASA and AAS prove **congruence**.

Coordinate tools

- Parallel lines have equal slopes; perpendicular slopes multiply to -1.
- The **perpendicular bisector** of AB passes through the midpoint with the negative reciprocal slope. Every point on it is the same distance from A and B.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Reasons and proof

- A proof gives a reason that works for every case: "equal radii", "corresponding angles on parallel lines".
- Measuring one diagram does not prove a general result.
- In a construction, say why each step works, for example why arcs of equal radius find points equidistant from A and B.

Worked example: order matters

- Rotate: $(-y, x) = (3, 2)$. Translate: $(7, 3)$.
- Other order: translate to $(6, -2)$, then rotate to $(2, 6)$.
- The results differ, so the order of the transformations matters.

Worked example: a segment and two triangles

- Length: $\sqrt{6^2 + 8^2} = 10$. Midpoint: $(2, 6)$.
- Slope of AB: $\frac{8}{6} = \frac{4}{3}$, so the **perpendicular bisector** has slope $-\frac{3}{4}$: $y - 6 = -\frac{3}{4}(x - 2)$.
- Two triangles with angles 40°, 60°, 80° and longest sides 9 cm and 15 cm are similar, not congruent. Length ratio $\frac{15}{9} = \frac{5}{3}$; area ratio $\frac{25}{9}$.

Common errors

- Using $(y, -x)$ for an anticlockwise rotation (that is clockwise).
- Assuming two transformations can be done in either order.
- Calling triangles congruent because their angles are equal.

Common errors (2)

- Using the same slope, not the negative reciprocal, for a perpendicular line.
- Treating a measurement on one diagram as a proof.

Key terms

- **rigid motions**: Transformations that keep lengths and angles: translations, reflections and rotations.
- **translation**: A slide of every point by the same vector.
- **dilation**: An enlargement or reduction by a scale factor about a centre.
- **similarity**: Same shape, with lengths in a fixed ratio.

Check yourself

- Reflect $(3, -4)$ in the x-axis.
- Lines have slopes 2 and $-\frac{1}{2}$. Are they parallel, perpendicular or neither?
- Find the midpoint and length of the segment from $(0, 0)$ to $(8, 6)$.