Functions, composition, transformations and inverses

Original skill preparation; versioned source scope

Functions: Composition, Transformations and Inverses

- Find the **domain 定义域** and **range 值域** of a function.
- Form a **composition 函数复合** and find its domain.
- Describe a **transformation 变换** of a graph.
- Find an **inverse function 反函数**, which needs a **one-to-one 单射** function.

Notation, domain and range

- $f(3)$ means the output when the input is 3.
- The domain excludes inputs that divide by zero or take an even root of a negative number.
- The range is the set of outputs; a graph or a vertex often shows it.

Composition

- $f(g(x))$ means: apply g, then apply f to the result.
- The domain of $f(g(x))$ contains only inputs where g is defined and $g(x)$ is allowed in f.
- Usually $f(g(x)) \neq g(f(x))$.

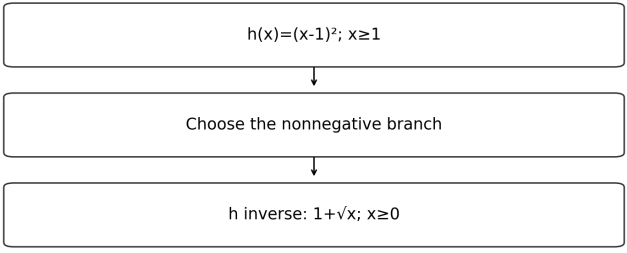
Transformations

- $f(x)+k$: up k. $f(x-h)$: right h. $af(x)$: vertical stretch by a, reflected if $a<0$. $f(-x)$: reflection in the y-axis.
- Track a point: if $(1,1)$ is on $y=x^2$, then on $y=3(x+4)^2-2$ it moves to $(-3,1)$.

Inverse functions

- An inverse undoes a function: if $f(a) = b$, then $f^{-1}(b) = a$.
- Only a one-to-one function, which passes the horizontal-line test, has an inverse. Otherwise, restrict the domain first.
- To find it: write $y = f(x)$, solve for x, then swap the letters. The domain and range swap too.

Figure: Inverse functions



- Worked steps: h(x)=(x-1)^2; x$\geq$1 $\rightarrow$ Choose the nonnegative branch $\rightarrow$ h inverse: 1+$\sqrt{}$x; x$\geq$0
- Match each part of the diagram to the text.

Worked example: composition both ways

- $f(g(x)) = \dfrac{1}{x^2 + 3 - 3} = \dfrac{1}{x^2}$, with $x \neq 0$.
- $g(f(4))$: $f(4) = 1$, then $g(1) = 4$.
- $f(g(4)) = \dfrac{1}{16}$: a different answer, because the order changed.

Worked example: an inverse on a branch

- On this domain $x+2 \leq 0$, so solving $y=(x+2)^2$ gives $x+2=-\sqrt{y}$.
- $h^{-1}(x)=-2-\sqrt{x}$, with domain $x \geq 0$ and range $y \leq -2$.
- The $+$ branch, $-2+\sqrt{x}$, gives values above -2, outside the original domain.

Common errors

- Applying f before g in $f(g(x))$.
- Ignoring the domain of the inner function.
- Shifting $f(x+4)$ to the right instead of the left.

Common errors (2)

- Finding an inverse for a function that is not one-to-one.
- Choosing the wrong square-root branch.

Key terms

- **domain**: The set of allowed inputs.
- **range**: The set of possible outputs.
- **composition**: Applying one function to the output of another.
- **transformation**: A shift, stretch or reflection of a graph.

Check yourself

- Find the inverse of $f(x) = 2x - 6$.
- Is $f(x) = x^2$, on all real numbers, one-to-one?
- Write $y = \sqrt{x}$ shifted 3 left and 1 up.