Real numbers, rational powers and complex arithmetic

Original skill preparation; versioned source scope

Real Numbers, Rational Powers and Complex Arithmetic

- Classify numbers as a **rational number** 有理数 or an **irrational number** 无理数.
- Use the rules for integer and rational powers, and simplify roots.
- Calculate with the **imaginary unit** 虚数单位 i and complex numbers.
- Divide complex numbers using the **complex conjugate** 复共轭.

Rational and irrational numbers

- A **rational number** can be written as a fraction of integers; its decimal ends or repeats.
- $\sqrt{2}$, $\sqrt{3}$ and π are irrational.
- Sums and products of irrational numbers can be rational: $\sqrt{2} \cdot \sqrt{2} = 2$ and $\sqrt{2} + (-\sqrt{2}) = 0$. So test examples instead of assuming.

Powers and roots

- A negative power gives a reciprocal, not a negative number: $2^{-3} = \frac{1}{8}$.
- Simplify a root by taking out square factors: $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.
- $\sqrt{x^2} = |x|$, so $\sqrt{(-7)^2} = 7$.

$$x^{-n} = \frac{1}{x^n} \qquad x^{m/n} = \left(\sqrt[n]{x}\right)^m \qquad x^a x^b = x^{a+b}$$

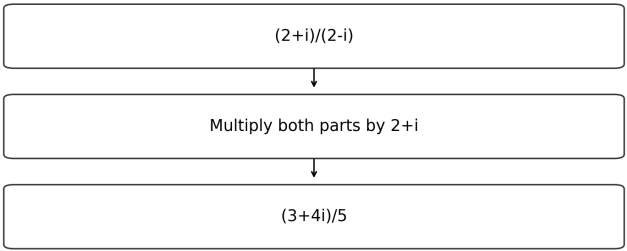
Complex numbers

- $i^2 = -1$, and powers of i repeat every four: $i, -1, -i, 1$.
- A complex number $a + bi$ has real part a and imaginary part b.
- Add by combining like parts; multiply like brackets and replace i^2 with -1.
- $z^2 = -16$ has the two solutions $z = 4i$ and $z = -4i$.

Dividing with the conjugate

- The conjugate of $a + bi$ is $a - bi$, and $(a + bi)(a - bi) = a^2 + b^2$, a real number.
- To divide, multiply the top and bottom by the conjugate of the bottom.

Figure: Dividing with the conjugate



- Worked steps: (2+i)/(2-i) → Multiply both parts by 2+i → (3+4i)/5
- Match each part of the diagram to the text.

Worked example: a quotient

- Multiply by $\dfrac{1-i}{1-i}$.
- Top: $(3-2i)(1-i) = 3-3i-2i+2i^2 = 1-5i$. Bottom: $1+1=2$.
- Answer: $\frac{1}{2} - \frac{5}{2}i$.
- Check: $(\frac{1}{2} - \frac{5}{2}i)(1+i) = \frac{1}{2} + \frac{1}{2}i - \frac{5}{2}i + \frac{5}{2} = 3-2i$.

Worked example: classifying and evaluating

- $\sqrt{50} = 5\sqrt{2}$: irrational. $\sqrt{50} \cdot \sqrt{2} = \sqrt{100} = 10$: rational.
- $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$: irrational. So "irrational times irrational" can go either way.
- $32^{2/5} = (\sqrt[5]{32})^2 = 2^2 = 4$, and $2^{-3} = \frac{1}{8}$.

Common errors

- Treating a negative power as a negative number.
- Writing $\sqrt{(-7)^2} = -7$.
- Forgetting that $i^2 = -1$ when multiplying.

Common errors (2)

- Dividing complex numbers by splitting the denominator.
- Assuming the product of two irrational numbers is irrational.

Key terms

- **rational number**: A number that can be written as a fraction of two integers.
- **irrational number**: A real number that cannot be written as such a fraction.
- **imaginary unit**: The number i, with $i^2 = -1$.
- **complex conjugate**: For $a + bi$, the number $a - bi$.

Check yourself

- Find i^7.
- Simplify $\sqrt{72}$.
- Add $(2+i)+(3-4i)$, then multiply $(1+i)(1-i)$.