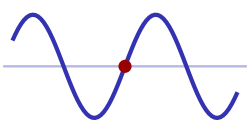
Oscillations

A2 Physics ▪ Topic 17

A-Level Physics



What we will cover

1. Simple harmonic motion 17.1
2. Proving SHM; phase 17.1.1
3. Energy in SHM 17.2
4. Damping and resonance 17.3
5. Recap



1 Simple harmonic motion

Sheet 17.1 – what you need to know

You must be able to:

- state the defining equation $a = -\omega^2 x$ and what it means
- use $x = x_0 \sin \omega t$, $v = v_0 \cos \omega t$, $v = \pm\omega\sqrt{x_0^2 - x^2}$
- read T, ω and x_0 off x–t, v–t, a–x and E_K–t graphs
- sketch v against x

What the exam asks

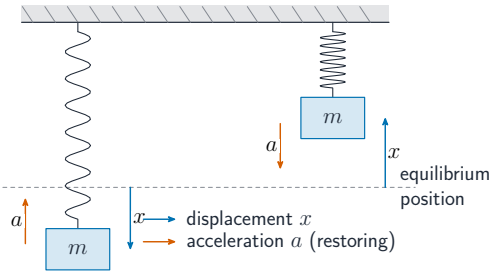
"Explain how Fig. 5.2 shows that the oscillations of the ball are simple harmonic." **[2]**

"On Fig. 5.1, sketch the variation of v with x." **[3]**

"Explain how Fig. 4.2 shows that the period of the oscillations is $0.80\,\mathrm{s}$." **[1]**

Practise on exercise sheet 17.1, then try the real exam questions on past-paper sheet 17.1.

Simple harmonic motion



A body moves with **simple harmonic motion** 简谐运动 when its acceleration is

- **proportional to its displacement** from a fixed **equilibrium** 平衡 point, and
- **directed back towards that point**.

Many systems do this near a stable equilibrium: a mass on a **spring** 弹簧, a **pendulum** 单摆 (small swing), a floating block, the charge on a **capacitor** 电容器 in an LC circuit, atoms in a solid.

The words you must use precisely

displacement x – from equilibrium, signed

amplitude x_0 – the largest $|x|$, always $+$

period T – one full oscillation

frequency $f = 1/T$, in Hz

angular frequency $\omega = 2\pi f = 2\pi/T$

phase difference – a fraction of a cycle, in rad

Given any one of ω, f, T you can reach the other two – start every question by getting ω.

Worked example – maximum acceleration

A mass oscillates with amplitude $0.050\,\mathrm{m}$ at $2.5\,\mathrm{Hz}$. Find its maximum acceleration.

$\omega = 2\pi f = 2\pi \times 2.5 = 15.7\,\mathrm{rad\,s^{-1}}$

$|a|$ is largest at the extremes, where $|a| = \omega^2 x_0$:

$a_{\max} = 15.7^2 \times 0.050 \approx 12\,\mathrm{m\,s^{-2}}$

Fastest at the centre, most accelerated at the ends – the two extremes are opposite. Confusing them is the classic error.

Displacement, velocity, acceleration

Starting from $x = 0$ moving positive:

$$x = x_0 \sin \omega t$$

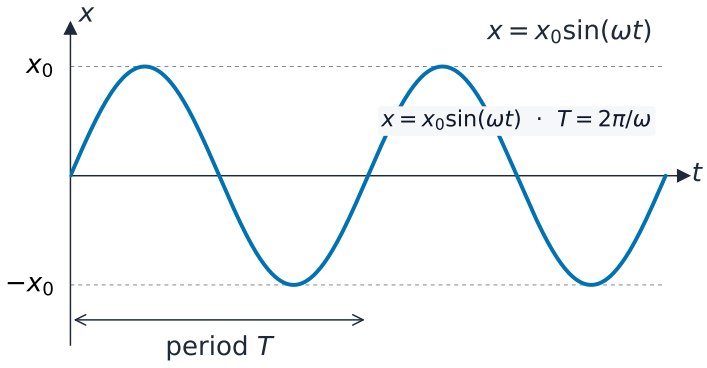
$$v = x_0 \omega \cos \omega t = v_0 \cos \omega t$$

$$a = -x_0 \omega^2 \sin \omega t = -\omega^2 x$$

so $v_0 = x_0 \omega$ and $a_{\max} = \omega^2 x_0$.

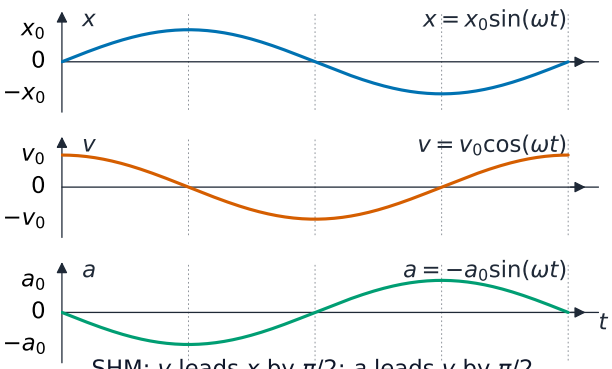
> Starting at the extreme instead? Use $x = x_0 \cos \omega t$. Pick the one that matches $t = 0$.

Displacement, velocity, acceleration, in detail



Displacement is a sine curve of amplitude x_0 when the motion starts at the centre, and a cosine when it starts at the extreme. Pick the one that matches $t = 0$.

The three graphs together

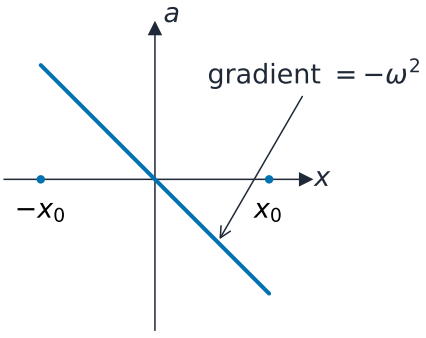


SHM: v leads x by $\pi/2$; a leads v by $\pi/2$

- x: sine, amplitude x_0
- v: cosine, leads x by $\pi/2$, amplitude ωx_0
- a: $-$sine, π out of phase with x, amplitude $\omega^2 x_0$

Each differentiation advances the phase by $\pi/2$ and multiplies the amplitude by ω – that is the whole pattern.

Reading ω off the a–x graph



$a = -\omega^2 x$ is a straight line through the origin with gradient $-\omega^2$. So

$$\omega = \sqrt{|\text{gradient}|}, \qquad T = \frac{2\pi}{\omega}$$

A straight line through the origin with negative gradient *is* the test for SHM – and a very common exam route to T.

SHM: $a = -\omega^2 x$ · gradient $= -\omega^2$

Velocity without time

A useful relation that does not use *time*:

$$v = \pm\omega\sqrt{x_0^2 - x^2}$$

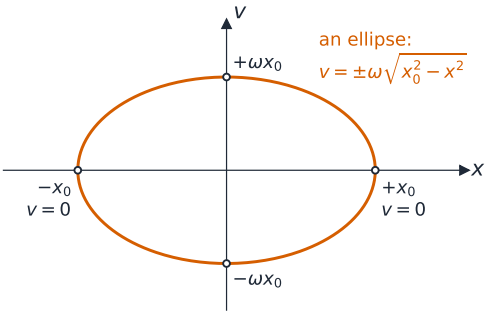
- at $x = 0$: $v = \pm\omega x_0$ – the maximum speed
- at $x = \pm x_0$: $v = 0$ – at rest for an instant

The $\pm$ is real: the particle crosses equilibrium twice a cycle, once each way.

$x_0 = 0.050\,\mathrm{m}$, $\omega = 15.7\,\mathrm{rad\,s^{-1}}$. Find v at $x = 0.030\,\mathrm{m}$.

$$v = 15.7\sqrt{0.050^2 - 0.030^2}$$
$$= 15.7 \times 0.040 \approx 0.63\,\mathrm{m\,s^{-1}}$$

Sketching v against x



Three marks: a closed ellipse; $v = 0$ at $x = \pm x_0$; $v = \pm\omega x_0$ at $x = 0$.

The velocity of an oscillator is $v = 0.56\cos(16t)$, in SI units. Find T and the amplitude x_0.

$$\omega = 16\,\mathrm{rad\,s^{-1}}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{16} = 0.39\,\mathrm{s}$$

$$v_0 = \omega x_0 \;\Rightarrow\; x_0 = \frac{v_0}{\omega} = \frac{0.56}{16} = 0.035\,\mathrm{m}$$

Read ω and v_0 straight off the equation: never put a time into it to find T.

Reading SHM from a graph: answers that score

"Explain how the a–x graph shows the motion is simple harmonic." [2]

A straight line through the origin, so $a \propto x$; a negative gradient, so a is always opposite to x.

"Define frequency" [1]: the number of oscillations per unit time.

"Explain how the E_K–t graph shows that the period is $0.80\,\text{s}$." [1]

E_K peaks every $0.40\,\text{s}$. It peaks twice each cycle (passing equilibrium each way), so $T = 2 \times 0.40 = 0.80\,\text{s}$.

"State two times at which the sphere passes through equilibrium in the *same* direction." [1]

Two crossings one whole period apart (e.g. t_1 and $t_1 + T$), not half a period: those go opposite ways.

2

Proving SHM

Sheet 17.1.1 – what you need to know

You must be able to:

- show a system is SHM in four steps: displace, restoring force, $a = F/m$, compare with $a = -\omega^2 x$
- derive ω for a spring, a floating body and a pendulum
- find a phase difference from a time lag: $\Delta\phi = 2\pi\Delta t/T$
- find g from the gradient of T^2 against L

What the exam asks

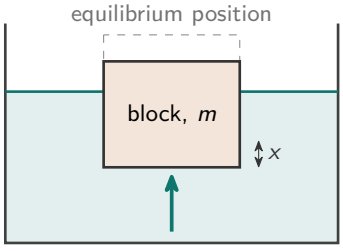
"State what is meant by simple harmonic motion." **[2]**

"State two times at which the sphere is passing in the same direction through the equilibrium position." **[1]**

"Use your answer in (b)(ii) and data from Fig. 4.2 to determine the maximum displacement of the sphere." **[3]**

Practise on exercise sheet 17.1.1, then try the real exam questions on past-paper sheet 17.1.

Show that the motion is SHM – always four steps



The algebra is never the mark on its own: step 4 must say ω^2 is constant.

Spring: $a = -\frac{k}{m}x$. Pendulum, small angle ($\sin\theta \approx \theta$): $a = -\frac{g}{L}x$.

A floating block (area A, mass m) is pushed down by x and released. Show that it moves with SHM.

1 Displace by x: extra volume submerged $= Ax$.

2 Restoring force = extra upthrust: $F = -\rho g A\, x$

3 $a = \dfrac{F}{m} = -\dfrac{\rho g A}{m}\, x$

4 Compare $a = -\omega^2 x$: $\omega^2 = \dfrac{\rho g A}{m}$ is constant, so $a \propto -x$: SHM.

Phase difference from a time lag, and g from a pendulum

$$\Delta\phi = 2\pi \frac{\Delta t}{T}$$

Δt = time lag between matching points

Two pendulums of period $2.0\,\mathrm{s}$ pass equilibrium $0.25\,\mathrm{s}$ apart. Find the phase difference.

$$\Delta\phi = 2\pi \times \frac{0.25\,\mathrm{s}}{2.0\,\mathrm{s}} = \frac{\pi}{4}\,\mathrm{rad}\ (0.79\,\mathrm{rad})$$

v leads x by $\frac{\pi}{2}$; a is π out of phase with x.

A plot of T^2 against L is a straight line of gradient $4.0\,\mathrm{s^2\,m^{-1}}$. Find g.

$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow T^2 = \frac{4\pi^2}{g} L$$

$$\text{gradient} = \frac{4\pi^2}{g}$$

$$g = \frac{4\pi^2}{4.0} = 9.9\,\mathrm{m\,s^{-2}}$$

The bob's mass cancels: it does not change T. The derivation assumes small angles and a light string.

3 Energy in SHM

Sheet 17.2 – what you need to know

You must be able to:

- use $E = \frac{1}{2} m\omega^2 x_0^2$ for the total energy
- sketch E_K and E_P against x and against t
- describe the interchange of kinetic and potential energy

What the exam asks

"On Fig. 5.4, sketch the variation with h of the potential energy E_P of the oscillations." **[3]**

"Calculate the total energy E of the vibrations." **[3]**

"Describe, without calculation, the interchange between the potential energy and the kinetic energy of the oscillations." **[3]**

Practise on exercise sheet 17.2, then try the real exam questions on past-paper sheet 17.2.

Energy in SHM

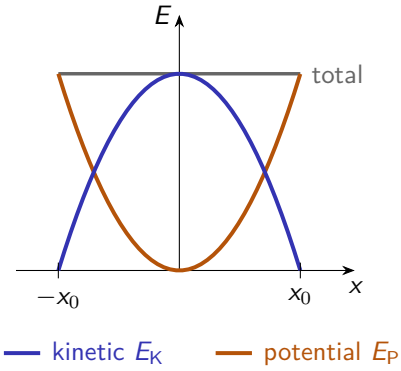
Energy swaps between **kinetic energy** 动能 and potential energy (a **spring** 弹簧's elastic store):

$$E_\mathrm{K} = \tfrac{1}{2} m\omega^2 (x_0^2 - x^2)$$
$$E_\mathrm{P} = \tfrac{1}{2} m\omega^2 x^2$$

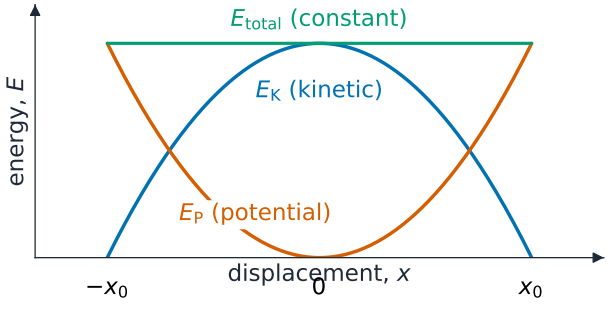
$$E_\mathrm{total} = \tfrac{1}{2} m\omega^2 x_0^2$$

conservation of energy 能量守恒

$E \propto x_0^2$: double the amplitude, four times the energy.



Energy in SHM, in detail



SHM: $E_k + E_p =$ const $\cdot$ max E_k at centre

Kinetic and potential energy swap over a cycle while the total energy stays constant

Worked example – total energy

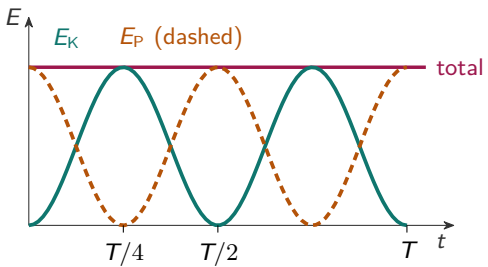
A $0.20\,\mathrm{kg}$ mass oscillates with $x_0 = 0.050\,\mathrm{m}$ and $\omega = 15.7\,\mathrm{rad\,s^{-1}}$. Find the total energy.

The total equals the maximum KE, as it passes $x = 0$ at $v_{\max} = \omega x_0$:

$$E = \tfrac{1}{2} m\omega^2 x_0^2 = \tfrac{1}{2}(0.20)(15.7)^2(0.050)^2$$
$$\approx 0.062\,\mathrm{J}$$

Evaluate the total where one form vanishes – at $x = 0$ it is all kinetic, at $x = x_0$ all potential. Either gives the same number.

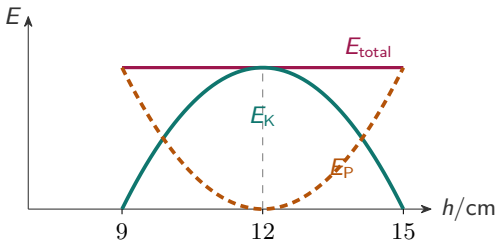
Energy against time: twice the frequency



- E_K and E_P each repeat every $T/2$: twice the oscillation frequency
- they are never negative and always add to the constant total
- with light damping the peaks stay equally spaced but shrink

Energy peaks $0.40\,\mathrm{s}$ apart mean the oscillation period is $0.80\,\mathrm{s}$.

Potential energy about an offset equilibrium



$E_P = E_{total} - E_K$: a parabola with its minimum at equilibrium ($12\,\text{cm}$), not at $h = 0$. The amplitude is half the range of h: $3\,\text{cm}$.

"Describe, without calculation, the interchange between the potential and kinetic energy." [3]

At the extremes all the energy is potential; at equilibrium it is all kinetic. In between one turns into the other, and the total stays constant.

$v = \pm 9.7\sqrt{11.6 - x^2}$ (v in cm s^{-1}, x in cm). Find f and x_0.

$\omega = 9.7\,\text{rad s}^{-1}$, so $f = \dfrac{\omega}{2\pi} = 1.5\,\text{Hz}$

$x_0^2 = 11.6$, so $x_0 = 3.4\,\text{cm}$

4 Damping and resonance

Sheet 17.3 – what you need to know

You must be able to:

- explain damping as energy lost to a resistive force, so the amplitude falls
- tell light, critical and heavy damping apart and sketch each
- state the conditions for resonance and how damping changes the curve

What the exam asks

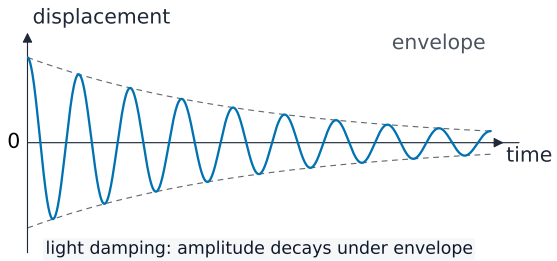
"By reference to electromagnetic induction and to conservation of energy, explain why the oscillations are damped." **[3]**

"Describe what is observed to indicate that the damping is light." **[1]**

"Sketch the variation of the displacement x of the sphere with t from $t = 0$ to $t = 6.0\,\mathrm{s}$." **[3]**

Practise on exercise sheet 17.3, then try the real exam questions on past-paper sheet 17.3.

Damped oscillations

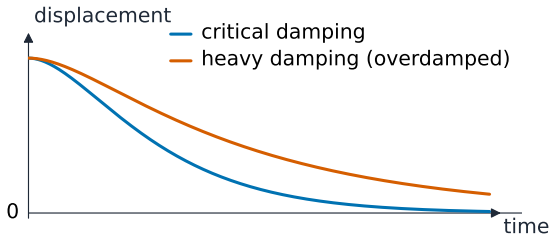


light damping: amplitude decays under envelope

A resistive force (**air resistance** 空气阻力, **friction** 摩擦力) removes energy as heat, so the amplitude shrinks. **Light damping** 轻阻尼: the amplitude dies away slowly over many cycles, still near the natural frequency.

> A car's suspension is light-to-medium damped – bumps die away, but the ride stays smooth.

Critical and heavy damping

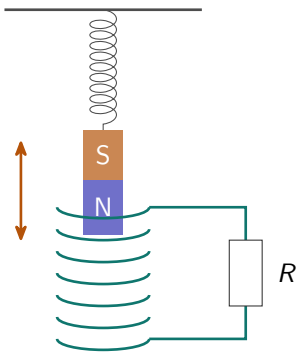


critical: fastest no-oscillation return · heavy: overdamped

- **critical damping** 临界阻尼: back to equilibrium in the shortest time, no overshoot, no oscillation
- **heavy damping** 过阻尼: no oscillation either, but slower than critical

A **galvanometer** 检流计 needle is critically damped so it settles at once; a strong door closer is heavily damped.

Electromagnetic damping



"By reference to electromagnetic induction and to conservation of energy, explain why the oscillations are damped." [3]

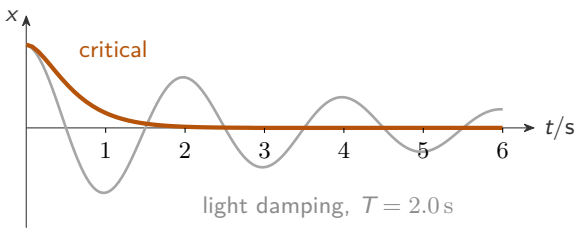
The moving magnet changes the flux linkage of the coil, so an e.m.f. is induced.

The e.m.f. drives a current through R, which heats it.

That thermal energy comes from the oscillation, so its amplitude falls.

A larger R gives a smaller current, less heating, so less damping. Light damping is seen as many oscillations whose amplitude falls gradually.

Sketching critical damping on given axes



light damping, $T = 2.0\,\mathrm{s}$

What earns the three marks:

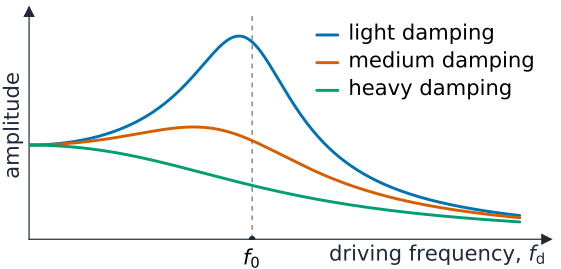
- starts at the same x_0, flat at $t = 0$
- falls to zero in well under one period
- never crosses the axis, then stays on it

> Heavy damping also never crosses, but takes longer to reach zero.

Forced oscillations and resonance

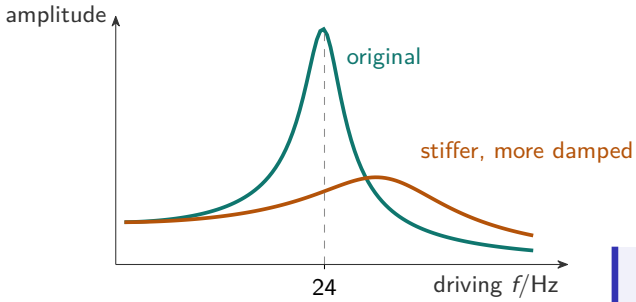
A **forced oscillation** 受迫振动 runs at the driving frequency f_d, not its own. **Resonance** 共振: $f_d = f_0$, the **natural frequency** 固有频率 – largest amplitude, most efficient transfer.

> Damping shapes the **resonance curve** 共振曲线: lighter → sharper and higher; heavier → broader, lower, and nudged to a lower frequency.



resonance: max amplitude near f_0 · lighter damping ⇒ sharper peak

Comparing resonance curves

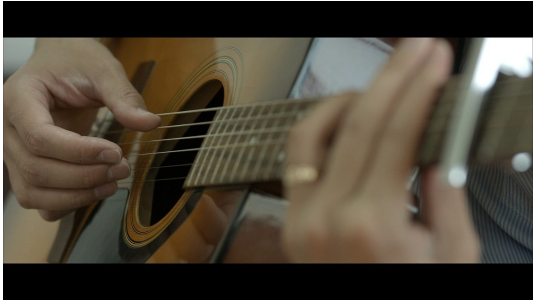


A machine vibrates at $24\,\mathrm{Hz}$; its mounting resonates there.

- stiffer: higher natural frequency, so the peak moves right
- more damping: the peak is lower and broader
- at $24\,\mathrm{Hz}$ the new amplitude is far smaller: no resonance

Resonance: driving frequency = natural frequency, giving maximum amplitude.

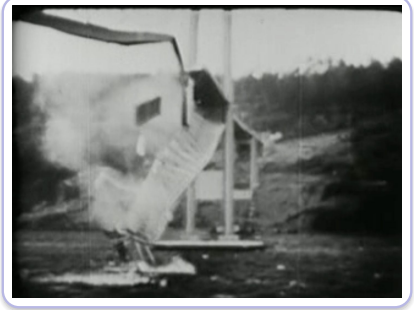
Resonance in the world



- a swing pushed at the right rate builds a big amplitude
- a wine glass shattered by its own ringing note
- a building shaken by an earthquake at a matching frequency

Engineers design a building's natural frequencies to avoid the main earthquake range – resonance is as often something to dodge as to exploit.

Resonance that brought a bridge down



Tacoma Narrows, 1940: a driving frequency close to a natural frequency, with too little damping to limit the amplitude.

5 Recap

Exam tips

- The **SHM condition is** $a = -\omega^2 x$ (acceleration proportional to displacement, directed back to equilibrium).
- Learn $x = x_0 \sin \omega t$ (or cos), $v_{max} = \omega x_0$, $a_{max} = \omega^2 x_0$; KE and PE interchange with the **total energy constant**.
- Velocity is **zero at the extremes** and maximum at the centre.
- **Resonance** occurs when the driving frequency equals the natural frequency; damping lowers and broadens the peak.

Topic 17 – key takeaways

1 The condition

$a = -\omega^2 x$ – proportional, and back to equilibrium

2 Maxima

$v_{\max} = \omega x_0$ at the centre;
$a_{\max} = \omega^2 x_0$ at the ends

3 Energy

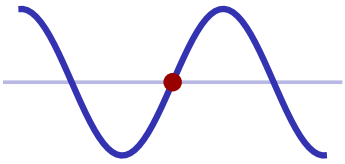
$E = \frac{1}{2} m\omega^2 x_0^2$, constant; $E \propto x_0^2$

4 Resonance

peak at $f_d = f_0$; damping lowers and broadens it

Check yourself

1. Where in the cycle is the speed greatest?
2. What is the gradient of an a–x graph?
3. Triple the amplitude – what happens to E?
4. Which damping settles fastest with no overshoot?



Answers **1.** at equilibrium, $x = 0$ **2.** $-\omega^2$ **3.** $\times 9$ **4.** critical