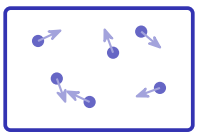
Ideal Gases

A2 Physics ▪ Topic 15

A-Level Physics



What we will cover

1. The mole 15.1
2. The equation of state 15.2
3. Kinetic theory 15.3
4. Deriving $pV = \frac{1}{3}Nm\langle c^2 \rangle$ 15.3.1
5. Recap

1 The mole

Sheet 15.1 – what you need to know

You must be able to:

- state what the Avogadro constant is and use $N = nN_A$
- find the mass of one molecule from the molar mass
- predict how p, n, $\langle c^2 \rangle$ and U change between samples

What the exam asks

"State what is meant by the Avogadro constant." **[1]**

"Complete Table 3.1 by giving expressions, in terms of some or all of N, m, T, V and the constants." **[4]**

"Sketch the variation with thermodynamic temperature of the r.m.s. speed of the molecules." **[2]**

Practise on exercise sheet 15.1, then try the real exam questions on past-paper sheet 15.1.

The mole

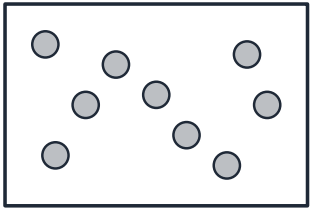
Amount of substance 物质的量 n is measured in **moles** 摩尔. One mole holds **Avogadro constant** 阿伏伽德罗常数 particles – **atoms** 原子 for a **monatomic** 单原子 gas like helium, molecules for O_2:

$$N_A = 6.02 \times 10^{23}\,\mathrm{mol}^{-1}$$

Number of particles $N = nN_A$. Molar mass in grams is the relative formula mass – $18\,\mathrm{g}$ of water is one mole.

Count particles with moles, not with 10^{23} on every line. n is the quantity the gas laws actually use.

One mole, counted



1 mole

$= 6.02 \times 10^{23}$ particles (N_A)

A mole is 6.02×10^{23} particles – the number chosen so that a mole of a substance weighs its relative molecular mass in grams.

> One mole is 6.02×10^{23} particles – whatever the particle is. The molar mass in grams equals the relative formula mass, so $18\,\mathrm{g}$ of water is exactly one mole.

The Avogadro constant, and the mass of one molecule

"State what is meant by the Avogadro constant." [1]

The number of particles in one mole of a substance:
$N_A = 6.02 \times 10^{23}\,\text{mol}^{-1}$.

$$N = nN_A, \qquad m = \frac{M}{N_A}$$

M: molar mass

Nitrogen has a molar mass of $0.028\,\text{kg mol}^{-1}$. Find the mass of one molecule.

One mole has mass M and holds N_A molecules:

$$m = \frac{M}{N_A} = \frac{0.028\,\text{kg mol}^{-1}}{6.02 \times 10^{23}\,\text{mol}^{-1}}$$
$$= 4.7 \times 10^{-26}\,\text{kg}$$

Convert the molar mass to kg mol^{-1} first: $28\,\text{g mol}^{-1}$ is $0.028\,\text{kg mol}^{-1}$.

Change the sample: what happens to each quantity?

Sample X: N molecules of mass m in volume V at T. Sample Y: $2N$ molecules of mass $2m$ in $2V$, at the same T.

quantity	**Y compared with X**	**because**
pressure p	the same	$p = NkT/V$: N and V both double
amount n	doubled	$n = N/N_{\mathrm{A}}$
$\langle c^2 \rangle$	halved	$\langle c^2 \rangle = 3kT/m$, from $\frac{1}{2}m\langle c^2 \rangle = \frac{3}{2}kT$
internal energy U	doubled	$U = \frac{3}{2}NkT$: twice the molecules, same mean KE

Write the formula with every symbol, then ask which symbols change. The mean KE depends on T only.

2 The equation of state

Sheet 15.2 – what you need to know

You must be able to:

- use $pV = nRT = NkT$, with T in kelvin
- define an ideal gas
- read N or n from the gradient of a pV–T graph
- sketch a cycle of changes on p–V axes

What the exam asks

"Identify the constant B." **[1]**

"Show that the r.m.s. speed of molecules of this gas is approximately $500\,\mathrm{m\,s^{-1}}$." **[3]**

"Sketch the variation with volume of pressure for the gas as the gas undergoes the three changes." **[3]**

Practise on exercise sheet 15.2, then try the real exam questions on past-paper sheet 15.2.

The equation of state

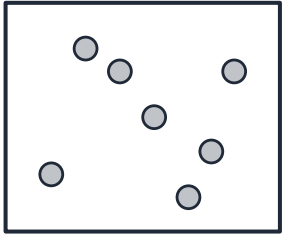
An **ideal gas** 理想气体 obeys $pV \propto T$ exactly:

$$pV = nRT \qquad \text{or} \qquad pV = NkT$$

- the **molar gas constant** 摩尔气体常数 $R = 8.31\,\mathrm{J\,mol^{-1}\,K^{-1}}$ – per mole
- the **Boltzmann constant** 玻尔兹曼常数 $k = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ – per molecule
- $N = nN_\mathrm{A}$, so $k = R/N_\mathrm{A}$

> Match the form to your amount: moles → nRT, molecules → NkT. SI throughout – Pa, m^3, and kelvin.

The equation of state, in detail



gas: p, V, T, n moles

$$pV = nRT$$

Every quantity in $pV = nRT$ is in SI units: pressure in Pa, volume in m^3, and temperature in kelvin – never °C.

Worked example – how many moles?

A $0.020\,\mathrm{m}^3$ cylinder holds gas at $27\,^\circ\mathrm{C}$ and $2.0\times10^5\,\mathrm{Pa}$. Find n. ($R = 8.31$)

First to kelvin: $T = 27 + 273 = 300\,\mathrm{K}$.

$$n = \frac{pV}{RT} = \frac{(2.0\times10^5)(0.020)}{8.31\times300} \approx 1.6\,\mathrm{mol}$$

The kelvin conversion is the first line, every time. 27 in place of 300 would inflate n by a factor of 11.

A fixed mass changing state

For a fixed amount of gas:

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

Hold one quantity fixed and a named law drops out.

Boyle

T fixed: pV constant

Charles

p fixed: V/T constant

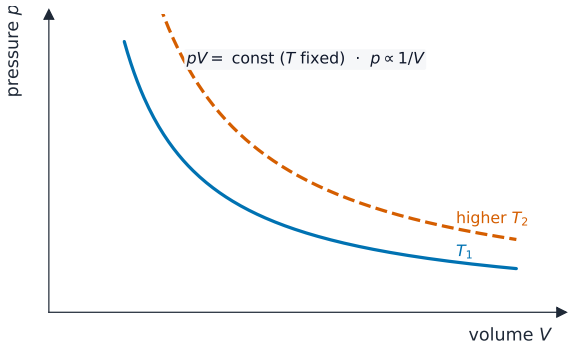
pressure law

V fixed: p/T constant

$pV \propto T$ holds only in kelvin – °C here is the classic lost mark.

Boyle's law

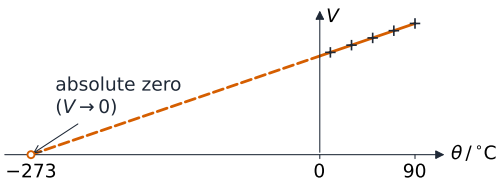
At constant temperature pV is constant, so p against $1/V$ is a straight line through the origin – the usual way an exam asks you to verify it.



Charles's law and absolute zero

At constant pressure the volume climbs linearly with temperature. Extrapolate the line back to zero volume and it meets $\approx -273\,^{\circ}\mathrm{C}$.

> Every gas points at the same intercept – which is what makes absolute zero a property of temperature, not of any one gas.



$V \propto T$ · Charles: extrapolates to absolute zero

Worked example – heating at constant pressure

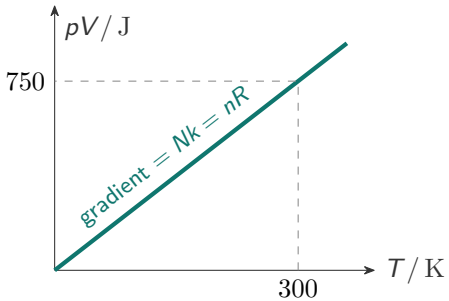
A fixed mass at $300\,\mathrm{K}$ fills $0.50\,\mathrm{m}^3$. Heat it to $450\,\mathrm{K}$ at constant pressure. Find the new volume.

p is fixed, so V/T is constant:

$$V_2 = V_1 \frac{T_2}{T_1} = 0.50 \times \frac{450}{300} = 0.75\,\mathrm{m}^3$$

Cancel what is constant before substituting – here p drops out and only the temperature ratio matters.

Reading a pV–T graph



A straight line through the origin: the gas is ideal.

The line passes through $(300\,\mathrm{K},\ 750\,\mathrm{J})$. Find the number of molecules N and the amount n.

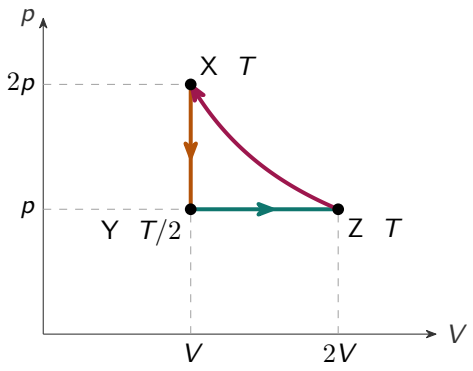
$$pV = NkT \Rightarrow N = \frac{\text{gradient}}{k} = \frac{750\,\mathrm{J}/300\,\mathrm{K}}{1.38\times10^{-23}\,\mathrm{J\,K^{-1}}}$$

$$= 1.8\times10^{23}$$

$$n = \frac{N}{N_\mathrm{A}} = \frac{1.8\times10^{23}}{6.02\times10^{23}\,\mathrm{mol^{-1}}} = 0.30\,\mathrm{mol}$$

Ideal gas: obeys $pV \propto T$ at all pressures, volumes and temperatures.

A gas taken round a cycle



Fixed mass, so $\frac{pV}{T}$ is constant. Start at X: $2p$, V, T.

- X→Y, constant V: p halves, so $T \to T/2$ – a vertical line
- Y→Z, constant p: V doubles, so $T/2 \to T$ – a horizontal line
- Z→X, constant T: pV stays $2pV$ – a curve, $p \propto 1/V$

Constant T is never a straight line on p–V axes: it is the curve $pV =$ constant.

3 Kinetic theory

Sheet 15.3 – what you need to know

You must be able to:

- state the assumptions of the kinetic model
- use $pV = \frac{1}{3}Nm\langle c^2\rangle$ and $p = \frac{1}{3}\rho\langle c^2\rangle$
- link mean KE to temperature: $\frac{1}{2}m\langle c^2\rangle = \frac{3}{2}kT$
- explain why $U \propto T$ for an ideal gas

What the exam asks

"Explain why the internal energy of an ideal gas is proportional to its thermodynamic temperature." **[2]**

"Suggest why a gas at the surface of a star is unlikely to be ideal." **[2]**

"Determine, to three significant figures, the temperature of the surface of the star." **[2]**

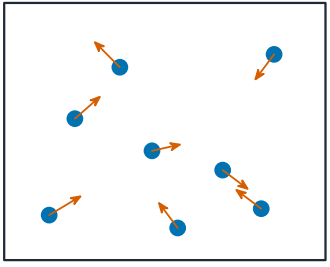
Practise on exercise sheet 15.3, then try the real exam questions on past-paper sheet 15.3.

Kinetic theory: the assumptions

1. many identical molecules in **random motion** 无规则运动
2. their own volume is negligible
3. collisions are brief; straight lines between
4. no **intermolecular** 分子间 forces except in a collision
5. collisions are **elastic** 弹性碰撞
6. Newton's laws apply

> Elastic is what keeps a gas from cooling itself. The model fails at high p (volume matters) and low T (forces matter).

Assumptions



1. many identical molecules, in continuous **random motion**
2. their own volume is **tiny** next to the container
3. straight lines between hits - intermolecular forces **ignored**
4. collisions are **elastic**: no kinetic energy is lost

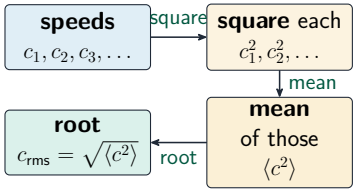
poor at very high pressure (volume matters) or very low temperature (forces matter)

The key assumptions of the kinetic theory of an ideal gas

Root-mean-square speed

r.m.s. speed 均方根 $= \sqrt{\langle c^2 \rangle}$: square each speed, average, then square-root. It sits slightly above the mean speed – squaring weights the fast molecules more.

> The gas needs $\langle c^2 \rangle$, not $\langle c \rangle$, because energy goes as c^2. Averaging the speeds first gives the wrong answer.



read the name **backwards**: **r**oot of the **m**ean of the **s**quares

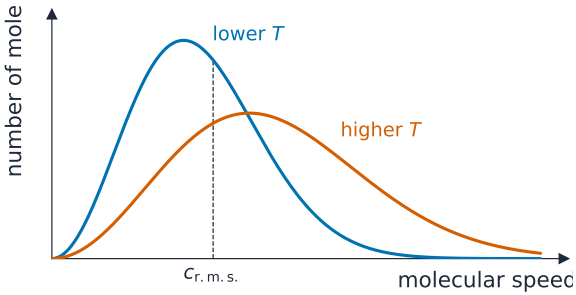
Why it matters under water

A diver's air is compressed: at $10\,\mathrm{m}$ the pressure is doubled, so a given mass of gas takes half the volume.

The spread of speeds

Molecules do not share one speed – they occupy a distribution. Raise the temperature and the curve broadens and shifts to faster speeds.

> The area is fixed (the molecules still all exist), so a broader curve must also be a lower one.



higher T: peak → higher speeds, flatter curve

The r.m.s. speed from pressure and density

Divide $pV = \frac{1}{3}Nm\langle c^2\rangle$ by V. The mass per unit volume Nm/V is the density ρ:

$$p = \tfrac{1}{3}\rho\langle c^2\rangle$$

No N, no m, no T needed: pressure and density are enough.

Air at $1.0 \times 10^5\,\mathrm{Pa}$ has a density of $1.2\,\mathrm{kg\,m^{-3}}$. Show that the r.m.s. speed of its molecules is about $500\,\mathrm{m\,s^{-1}}$.

$$c_{\mathrm{rms}} = \sqrt{\langle c^2\rangle} = \sqrt{\frac{3p}{\rho}}$$

$$= \sqrt{\frac{3 \times 1.0 \times 10^5\,\mathrm{Pa}}{1.2\,\mathrm{kg\,m^{-3}}}} = 500\,\mathrm{m\,s^{-1}}$$

Average translational KE

The **average translational kinetic energy** 平均平动动能, from two expressions for the same pV – the second using the **mean square** 均方 speed $\langle c^2 \rangle$:

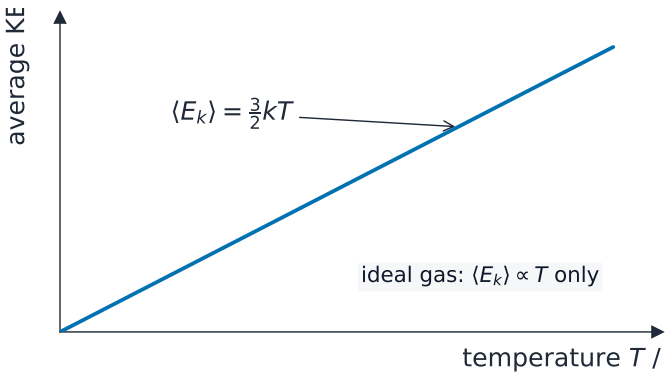
$$pV = NkT, \qquad pV = \tfrac{1}{3} Nm\langle c^2 \rangle$$

Set equal, cancel N, multiply by $\frac{3}{2}$:

$$\langle E_k \rangle = \tfrac{1}{2} m\langle c^2 \rangle = \tfrac{3}{2} kT$$

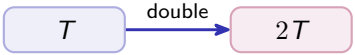
> The average molecular KE depends on temperature alone – not the gas, not the pressure. Squeezing a gas at fixed T gives more wall hits, not faster molecules.

Average translational KE, in detail



Mean molecular kinetic energy depends on T alone. Squeezing a gas at fixed T gives more wall hits, not faster molecules.

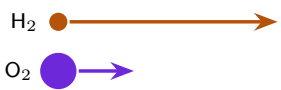
Consequences



$\langle c^2 \rangle$ **doubles**

$c_{\mathrm{r.m.s.}} \times \sqrt{2}$

same T



lighter $\Rightarrow$ faster on average

- **Mean KE** of a molecule $\propto T$ (absolute).
- Same T: same mean KE, so the **lighter** gas has the larger $\langle c^2 \rangle$.

$$E_{\mathrm{k}} = \tfrac{3}{2} NkT = \tfrac{3}{2} nRT$$

total translational KE of N molecules

Worked example – r.m.s. speed of oxygen

Find $c_{\text{r.m.s.}}$ for $\mathrm{O_2}$ at $300\,\mathrm{K}$.

$m = 5.3 \times 10^{-26}\,\mathrm{kg}$, $k = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$

From $\frac{1}{2}m\langle c^2\rangle = \frac{3}{2}kT$: $\langle c^2\rangle = 3kT/m$

$$c_{\text{r.m.s.}} = \sqrt{\frac{3(1.38 \times 10^{-23})(300)}{5.3 \times 10^{-26}}}$$

$\approx 480\,\mathrm{m\,s^{-1}}$

Faster than a jet – yet the gas goes nowhere, because the directions are random. Lighter gas, same $T \Rightarrow$ faster: hydrogen outruns oxygen in the same room.

Worked example – a temperature from an r.m.s. speed

Helium atoms (mass $6.6 \times 10^{-27}\,\mathrm{kg}$) have an r.m.s. speed of $1500\,\mathrm{m\,s^{-1}}$. Find the temperature of the gas.

Mean KE of one atom: $\frac{1}{2}m\langle c^2\rangle = \frac{3}{2}kT$

$$T = \frac{m\langle c^2\rangle}{3k} = \frac{6.6 \times 10^{-27}\,\mathrm{kg} \times (1500\,\mathrm{m\,s^{-1}})^2}{3 \times 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}}$$
$$= 360\,\mathrm{K}$$

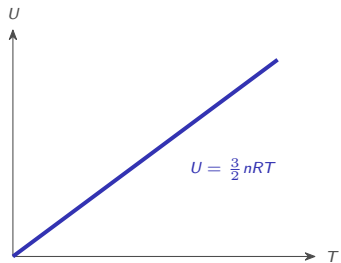
Square the r.m.s. speed to get $\langle c^2\rangle$. Then $T \propto \langle c^2\rangle$: doubling T multiplies c_{rms} by $\sqrt{2}$, not by 2.

Internal energy of an ideal gas

Ideal molecules are points: no intermolecular PE, no rotation or vibration in this model. So the **internal energy** 内能 is all kinetic:

$$U = \frac{3}{2}NkT = \frac{3}{2}nRT$$

> $U \propto T$ – double the thermodynamic temperature and you double the internal energy.



straight through the origin

Two explanations from the model

"Explain why the internal energy of an ideal gas is proportional to its thermodynamic temperature." [2]

There are no intermolecular forces, so no potential energy: U is the total kinetic energy of the molecules.

Mean KE $= \frac{3}{2}kT$, so $U = \frac{3}{2}NkT \propto T$.

"A gas is heated at constant volume. Explain why its pressure rises." [3]

The molecules move faster: each hits a wall with a larger change of momentum, and they hit the walls more often. So the force on the walls, and the pressure, increase.

Compressing at constant T is different: the molecules are not faster, they only hit the walls more often.

4 Deriving $pV = \frac{1}{3}Nm\langle c^2 \rangle$

Sheet 15.3.1 – what you need to know

You must be able to:

- derive $pV = \frac{1}{3}Nm\langle c^2\rangle$ in five steps
- tie each assumption to the step it allows
- state the meaning of N, m and $\langle c^2\rangle$
- show that $\frac{1}{2}m\langle c^2\rangle = \frac{3}{2}kT$

What the exam asks

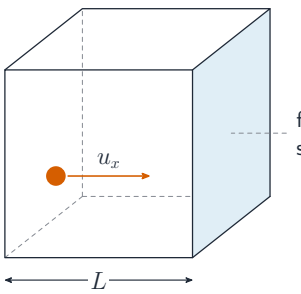
"State two of the basic assumptions of the kinetic theory of gases." **[2]**

"State the meaning of the symbols N, m and $\langle c^2\rangle$." **[3]**

"Use the equation of state to show that $E_K = \frac{3}{2}kT$." **[2]**

Practise on exercise sheet 15.3.1, then try the real exam questions on past-paper sheet 15.3.

Where the pressure comes from



One molecule, one wall, in a cube of side L:

- bounce: the **momentum** 动量 change is $\Delta p_x = -2mu_1$, so the wall takes an **impulse** 冲量 of $+2mu_1$
- next hit after $\Delta t = 2L/u_1$
- so $F_1 = \dfrac{\Delta p}{\Delta t} = \dfrac{mu_1^2}{L}$

Sum over N, then use $\langle u_x^2\rangle = \frac{1}{3}\langle c^2\rangle$:

$$pV = \tfrac{1}{3}Nm\langle c^2\rangle$$

The $\frac{1}{3}$ is symmetry: no direction is special, so each axis carries a third of the motion.

Each assumption has a job in the derivation

assumption	**the step it allows**
collisions are elastic	the molecule rebounds at the same speed: $\Delta p = 2mu$
molecules' own volume is negligible	each one crosses the whole length L: $\Delta t = 2L/u$
no forces between collisions	straight lines at constant speed between walls
collision time negligible	the time between hits is all travel time
large number, random motion	average over N, and no direction is special

$$c^2 = u^2 + v^2 + w^2 \;\Rightarrow\; \langle u^2\rangle = \tfrac{1}{3}\langle c^2\rangle$$

Pythagoras in 3D, then average

"State the meaning of $\langle c^2\rangle$": the mean square speed – the mean of the squared speeds, not the square of the mean.

5 Recap

Exam tips

- Use $pV = nRT$ (n in mol) or $pV = NkT$ (N molecules), with temperature in **kelvin**.
- Learn the **kinetic-theory assumptions** (random motion, negligible molecular volume, elastic collisions, no intermolecular forces).
- Mean translational KE $= \frac{3}{2}kT$ – it depends **only on temperature**.

Topic 15 – key takeaways

1 The mole

$N = nN_\mathrm{A}$, $N_\mathrm{A} = 6.02 \times 10^{23}$

2 State

$pV = nRT = NkT$, always in kelvin

3 Kinetic theory

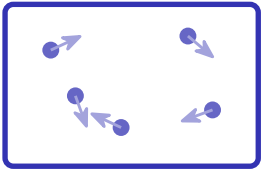
$pV = \frac{1}{3}Nm\langle c^2 \rangle$ from elastic bounces

4 Energy

$\langle E_\mathrm{k} \rangle = \frac{3}{2}kT$ – temperature only

Check yourself

1. How many molecules in $2.0\,\mathrm{mol}$?
2. Which form suits a count of molecules?
3. Why is there a $\frac{1}{3}$ in $pV = \frac{1}{3}Nm\langle c^2\rangle$?
4. At the same T, which is faster: $\mathrm{H_2}$ or $\mathrm{O_2}$?



Answers **1.** 1.2×10^{24} **2.** $pV = NkT$ **3.** symmetry – a third per axis **4.** $\mathrm{H_2}$