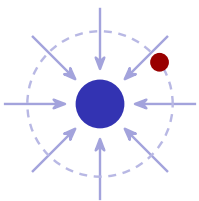
Gravitational Fields

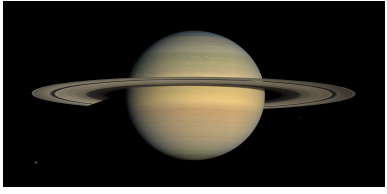
A2 Physics ▪ Topic 13

A-Level Physics



What we will cover

1. Gravitational field 13.1
2. Newton's law and orbits 13.2
3. Field of a point mass 13.3
4. Gravitational potential 13.4
5. Recap



1 Gravitational field

Sheet 13.1 – what you need to know

You must be able to:

- define a gravitational field and field strength $g = F/m$
- say what the direction of a field line shows
- describe the field lines of a planet and of a small patch near its surface
- explain why g is nearly constant for small heights

What the exam asks

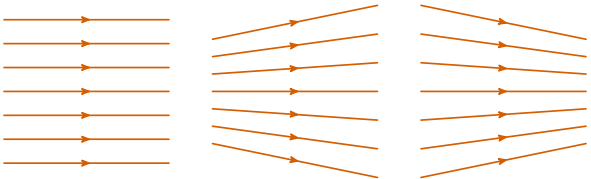
"Define gravitational field at a point." **[1]**

"Describe the pattern of the field lines outside the planet." **[2]**

"Explain why, for small changes in vertical height near the surface of the planet, g may be assumed to be constant." **[2]**

Practise on exercise sheet 13.1, then try the real exam questions on past-paper sheet 13.1.

Gravitational field strength

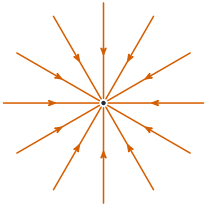


A **gravitational field** 重力场 is a region where a mass feels a force. Its **field strength** 重力场强度 is the force per unit mass on a small **test mass** 检验质量:

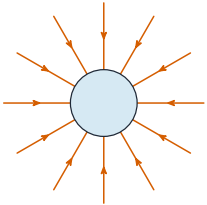
$$g = \frac{F}{m}, \qquad [g] = \mathrm{N\,kg^{-1}}$$

$\mathrm{N\,kg^{-1}}$ is $\mathrm{m\,s^{-2}}$ – g is also the acceleration of free fall. It is a vector, pointing at the source.

Radial and uniform fields



point mass



uniform sphere

- round a point mass or sphere: **radial** 径向, pointing inwards
- over a small patch of ground: parallel and evenly spaced – a **uniform field** 匀强场
- closer lines = stronger field

> Outside a uniform sphere the field is identical to a point mass of the same total mass at its centre – which is why you may treat the Earth as a point.

Field answers that score

"Define gravitational field." [1]

A region of space where a mass experiences a force.

"What does the direction of a field line show?" [1]

The direction of the force on a (small test) mass placed there.

"Describe the field lines outside a planet." [2]

Radial, pointing towards the centre; closer together nearer the surface.

"Why is g nearly constant near the surface?" [2]

Over a small height the lines are almost parallel and equally spaced: the height change is tiny compared with the planet's radius.

2 Newton's law and orbits

Sheet 13.2 – what you need to know

You must be able to:

- state Newton's law of gravitation and use $F = \dfrac{GMm}{r^2}$
- set gravity equal to the centripetal force for a circular orbit
- show that $4\pi^2 r^3 = GMT^2$, and use it on a graph
- describe a geostationary orbit

What the exam asks

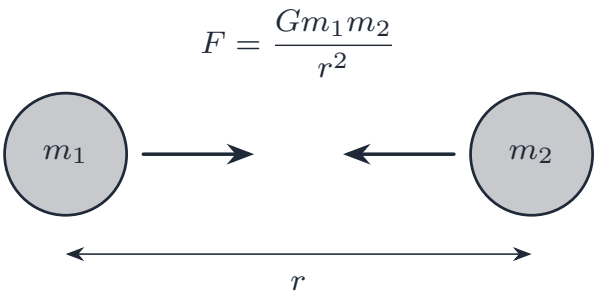
"State Newton's law of gravitation." **[2]**

"Use Newton's law of gravitation to show that R and T are related by $4\pi^2 R^3 = GMT^2$." **[3]**

"Use the gradient and intercept of the line to determine values for A and B." **[5]**

Practise on exercise sheet 13.2, then try the real exam questions on past-paper sheet 13.2.

Newton's law of gravitation



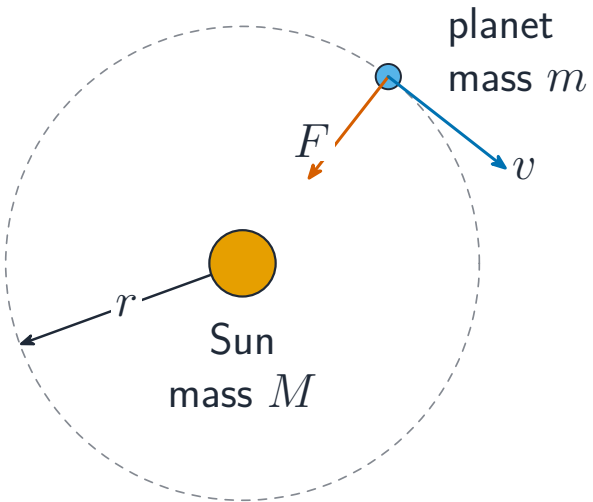
Two point masses attract along the line joining them:

$$F = \frac{Gm_1m_2}{r^2}$$

with
$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$.

Inverse square: double the separation and the force falls to a quarter, not a half.

Orbital motion



Gravity is the centripetal force:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$$

The satellite's mass cancels. A bolt and a space station at the same radius orbit at the same speed – which is why astronauts float alongside their craft.

Worked example – orbital speed

A satellite orbits Earth at $r = 7.0 \times 10^6\,\mathrm{m}$. Find its speed. ($GM = 4.0 \times 10^{14}\,\mathrm{m^3\,s^{-2}}$)

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{4.0 \times 10^{14}}{7.0 \times 10^6}}$$

$$\approx 7.6 \times 10^3\,\mathrm{m\,s^{-1}}$$

r is measured from the centre of the Earth, not the surface – a height above ground must have R added to it first.

Kepler's third law

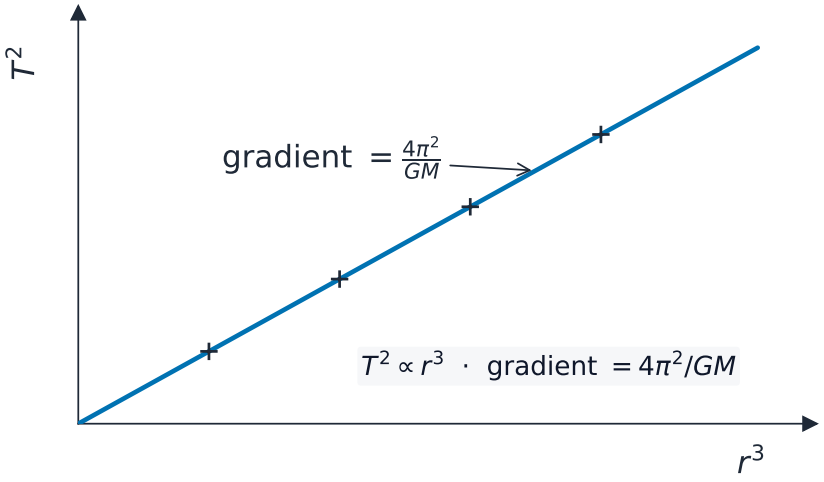
From $T = 2\pi r/v$:

$$T = 2\pi\sqrt{\frac{r^3}{GM}} \;\Rightarrow\; T^2 = \frac{4\pi^2}{GM}\,r^3$$

So $T^2 \propto r^3$ – a straight line through the origin.

> The gradient of T^2 against r^3 is $4\pi^2/(GM)$, so orbital data weighs the central body.

Kepler's third law, in detail



Kepler's third law: $T^2 \propto r^3$, a straight line through the origin

Orbit data on a straight-line graph

A satellite at height h has orbit radius $r = R + h$. Gravity is the centripetal force:

$$\frac{GMm}{(R+h)^2} = m(R+h)\frac{4\pi^2}{T^2}$$

$$\Rightarrow \quad (R+h)^3 = \frac{GM}{4\pi^2}T^2$$

Take cube roots:

$$T^{2/3} = \left(\frac{4\pi^2}{GM}\right)^{1/3}(h+R)$$

a straight line against h

A $T^{2/3}$–h line passes through $(0,\ 295\,\mathrm{s}^{2/3})$ and $(1.00\times10^7\,\mathrm{m},\ 757\,\mathrm{s}^{2/3})$. Find M and R.

$$k = \frac{757-295}{1.00\times10^7} = 4.62\times10^{-5}\,\mathrm{s}^{2/3}\,\mathrm{m}^{-1}$$

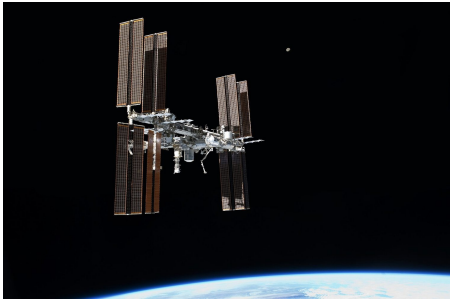
$$M = \frac{4\pi^2}{Gk^3} = \frac{4\pi^2}{6.67\times10^{-11}\times(4.62\times10^{-5})^3}$$

$$= 6.0\times10^{24}\,\mathrm{kg}$$

$$R = \frac{\text{intercept}}{k} = \frac{295}{4.62\times10^{-5}} = 6.4\times10^6\,\mathrm{m}$$

Geostationary orbit

It must have the same **angular speed** 角速度 as the Earth, in the same direction, in the equatorial plane – otherwise it drifts north–south during the day.



A **geostationary** 地球同步 satellite sits over one spot, so it must have:

- period $24\,\mathrm{h}$ – the Earth's own
- direction west to east
- plane: above the **equator** 赤道

Then $T^2 = 4\pi^2 r^3/(GM)$ fixes $r \approx 4.2 \times 10^7\,\mathrm{m}$.

> Off the equator it would drift north–south each day, so a fixed dish could not track it.

3 Field of a point mass

Sheet 13.3 – what you need to know

You must be able to:

- derive $g = \dfrac{GM}{r^2}$ from $F = \dfrac{GMm}{r^2}$ and $g = F/m$
- use it to find g, M or R
- compare the field at two distances, and sketch g against r
- sketch the field between two masses

What the exam asks

"By considering the force exerted by the point mass on a test mass, derive an equation for g at P." **[2]**

"Compare the gravitational field at Q with that at P." **[2]**

"Sketch the variation with x of the gravitational field g at point P." **[3]**

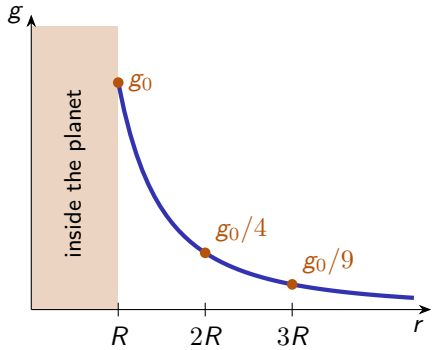
Practise on exercise sheet 13.3, then try the real exam questions on past-paper sheet 13.3.

Field strength from a point mass

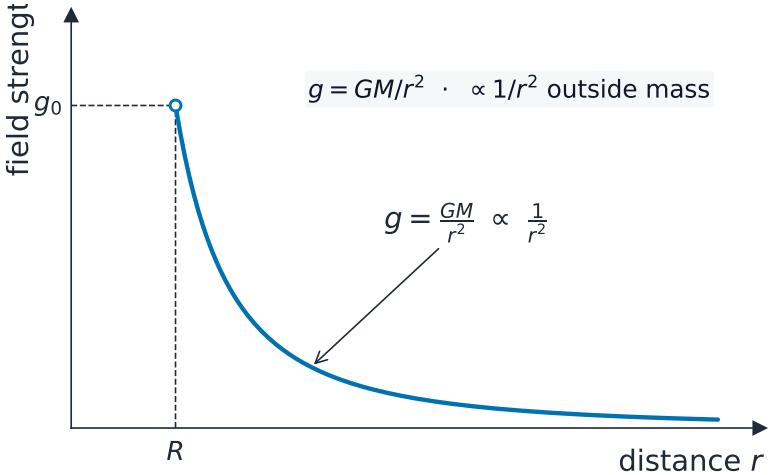
Put $F = GMm/r^2$ into $g = F/m$; the test mass cancels:

$$g = \frac{GM}{r^2}$$

g depends only on the source mass and the distance, never on the mass you put there.



Field strength from a point mass, in detail



Field strength falls off as $1/r^2$ with distance from a point mass

Worked example – g at the Earth's surface

Earth: $M = 6.0 \times 10^{24}\,\mathrm{kg}$, $R = 6.4 \times 10^{6}\,\mathrm{m}$. Find g at the surface.

$$g = \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11})(6.0 \times 10^{24})}{(6.4 \times 10^{6})^2}$$

$$\approx 9.8\,\mathrm{N\,kg^{-1}}$$

Why is g constant in the lab? Climbing $10\,\mathrm{m}$ changes r by one part in a million of $R = 6.4 \times 10^{6}\,\mathrm{m}$.

Worked examples – using $g = GM/r^2$

For the Earth $GM = 3.99 \times 10^{14}\,\mathrm{m^3\,s^{-2}}$ and $g = 9.81\,\mathrm{N\,kg^{-1}}$ at the surface. Find the radius R.

$$g = \frac{GM}{R^2} \;\Rightarrow\; R = \sqrt{\frac{GM}{g}}$$

$$R = \sqrt{\frac{3.99 \times 10^{14}\,\mathrm{m^3\,s^{-2}}}{9.81\,\mathrm{m\,s^{-2}}}}$$

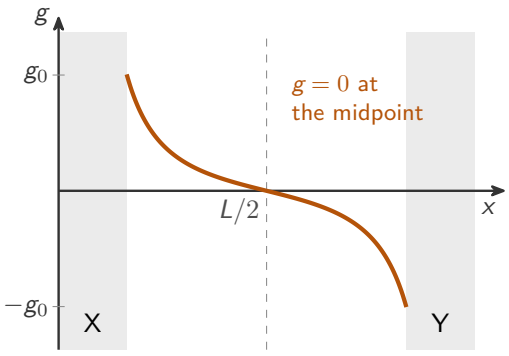
$$= 6.38 \times 10^6\,\mathrm{m}$$

P is x from a point mass. Q is $x/2$ from it, on the other side. Compare the fields.

$$\frac{g_Q}{g_P} = \frac{GM/(x/2)^2}{GM/x^2} = 4$$

Q's field is four times stronger and points the opposite way – both point towards the mass.

Between two equal spheres



Each sphere pulls towards itself, so on the line between them the two fields point in opposite directions:

- at the surface of X: $g = g_0$, towards X
- at the midpoint $L/2$: equal and opposite, $g = 0$
- at the surface of Y: $g = -g_0$ (towards Y)

Field is a vector: fields from two masses add as vectors, so they cancel at the midpoint. Potential is a scalar: there it is least negative, not zero.

4 Gravitational potential

Sheet 13.4 – what you need to know

You must be able to:

- define gravitational potential and use $\phi = -\dfrac{GM}{r}$
- explain why potential and potential energy are negative
- use $\Delta E_{\mathrm{P}} = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$ between two orbits
- find an escape speed

What the exam asks

"Define gravitational potential at a point." **[2]**

"Show that the mass of Mars is 6.4×10^{23} kg." **[3]**

"Explain why the gravitational potential energy of two point masses is always negative." **[2]**

Practise on exercise sheet 13.4, then try the real exam questions on past-paper sheet 13.4.

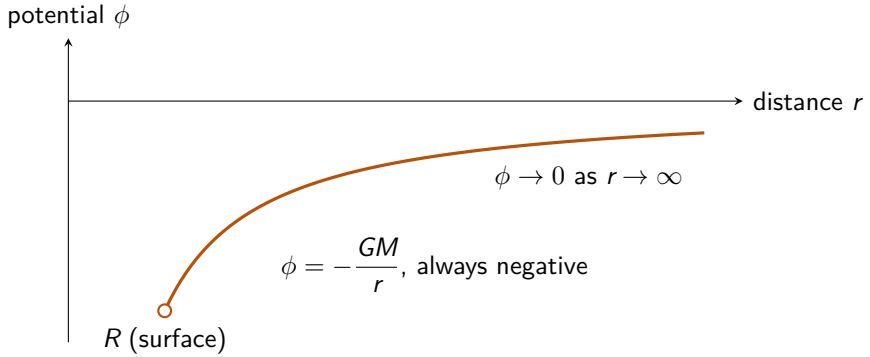
Gravitational potential

Potential 引力势 is the work done per unit mass bringing a test mass in from **infinity** 无穷远:

$$\phi = \frac{W}{m} = -\frac{GM}{r}, \qquad [\phi] = \mathrm{J\,kg^{-1}}$$

Zero at infinity, so ϕ is negative everywhere else – gravity did the work for you on the way in. ϕ is a scalar: just add them.

Gravitational potential, in detail



The **potential well**: $\phi = -GM/r$ is negative and rises to **zero at infinity**; escaping from the surface needs $\frac{1}{2}mv^2 = -m\phi$.

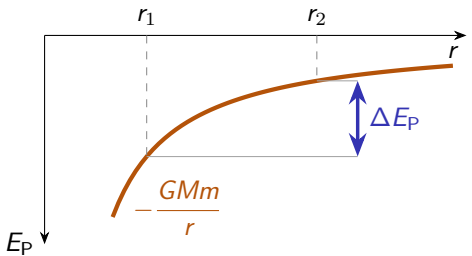
Link with $\Delta E_P = mg\Delta h$

Near the surface r barely changes, so g is constant:

$$\Delta E_P \approx mg\Delta h$$

For large changes (a satellite moving to a higher orbit) take the difference of $-GMm/r$.

$$\Delta E_P = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$



A small step on the curve is the straight $mg\Delta h$.

Potential energy and escape

A mass m sitting where the potential is ϕ:

$$E_{\mathrm{P}} = m\phi = -\frac{GMm}{r}$$

Raising a satellite from r_1 to r_2:

$$\Delta E_{\mathrm{P}} = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) > 0$$

To escape, the kinetic energy must match the well's depth:

$$\tfrac{1}{2}mv_{\mathrm{esc}}^2 = \frac{GMm}{r} \;\Rightarrow\; v_{\mathrm{esc}} = \sqrt{\frac{2GM}{r}}$$

$mg\Delta h$ is only the small-height shortcut, where r hardly changes. Over orbital distances you must difference $-GMm/r$ at each radius.

Escape needs no rocket left over: reach v_{esc} and you coast to infinity, arriving with zero speed.

The energy of two point masses

Potential energy	The **gravitational potential energy** 引力势能 of two point masses is $E_p = -\dfrac{GMm}{r}$ – negative, because the zero is placed at infinity.
The constant	$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} is the **universal gravitational constant** 万有引力常量 – the same everywhere, which is what "universal" asserts.
Escape speed	Escape needs $\frac{1}{2}mv^2 \geq \dfrac{GMm}{r}$, so $v_{\text{esc}} = \sqrt{2GM/r}$ – independent of the escaping mass.

The minus sign is not decoration: it says work must be *done* to separate the masses, and it is what makes a bound orbit have negative total energy.

Worked example – weighing a planet from an orbit change

A planet has radius $2.5 \times 10^6\,\mathrm{m}$. A $200\,\mathrm{kg}$ satellite is raised from $1.0 \times 10^6\,\mathrm{m}$ to $4.0 \times 10^6\,\mathrm{m}$ above the surface; its potential energy rises by $7.0 \times 10^8\,\mathrm{J}$. Find the planet's mass and the potential at its surface.

Radii from the centre: $r_1 = 3.5 \times 10^6\,\mathrm{m}$, $r_2 = 6.5 \times 10^6\,\mathrm{m}$

$$\Delta E_\mathrm{P} = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$M = \frac{\Delta E_\mathrm{P}}{Gm\,(1/r_1 - 1/r_2)} = 4.0 \times 10^{23}\,\mathrm{kg}$$

At the surface $r = R$:

$$\phi = -\frac{GM}{R} = -\frac{6.67 \times 10^{-11} \times 4.0 \times 10^{23}}{2.5 \times 10^6\,\mathrm{m}}$$

$$= -1.1 \times 10^7\,\mathrm{J\,kg^{-1}}$$

Add the radius to every height first. A potential needs its minus sign and the unit $\mathrm{J\,kg^{-1}}$.

Worked example – escape velocity

Find the escape velocity from the Earth's surface. ($GM = 4.0 \times 10^{14}\,\mathrm{m^3\,s^{-2}}$, $R = 6.4 \times 10^{6}\,\mathrm{m}$)

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2(4.0 \times 10^{14})}{6.4 \times 10^{6}}}$$

$$\approx 1.1 \times 10^{4}\,\mathrm{m\,s^{-1}} = 11\,\mathrm{km\,s^{-1}}$$

The mass cancels: a pebble and a rocket need the same $11\,\mathrm{km\,s^{-1}}$. Note the 2 – escape speed is $\sqrt{2}\times$ the circular orbital speed at the same radius.

5 Recap

Exam tips

- **Newton's law of gravitation** $F = GMm/r^2$ (inverse-square); field strength $g = GM/r^2$.
- Distinguish **gravitational potential** ($\phi = -GM/r$, always negative, zero at infinity) from field strength.
- For an orbit set gravity = centripetal force to get $T^2 \propto r^3$; a geostationary orbit has $T = 24$ h.

Topic 13 – key takeaways

1 Force

$F = GMm/r^2$ – inverse square, always attractive

2 Field

$g = GM/r^2$, a vector; the test mass cancels

3 Orbits

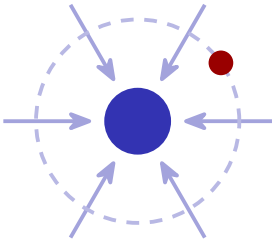
gravity = centripetal $\Rightarrow T^2 \propto r^3$

4 Potential

$\phi = -GM/r$, a scalar, zero at infinity

Check yourself

1. Triple the separation – what happens to F?
2. Why is ϕ always negative?
3. What is the period of a geostationary orbit?
4. Does escape velocity depend on the object's mass?



Answers **1.** falls to $1/9$ **2.** zero is set at infinity **3.** $24\,\mathrm{h}$ **4.** no – m cancels