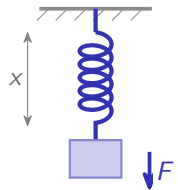
Deformation of Solids

AS Physics ▪ Topic 6

A-Level Physics



What we will cover

1. Hooke's law 6.1
2. The Young modulus 6.1.1
3. Elastic and plastic 6.2
4. Definitions and tips

1 Hooke's law

Sheet 6.1 – what you need to know

You must be able to:

- use the words tensile, compressive, load, extension and compression
- state Hooke's law and use $k = \dfrac{F}{x}$
- read a force–extension graph: its gradient, and where it stops being straight
- find the spring constant of springs in series or in parallel

What the exam asks

"State Hooke's law." **[1]**

"Determine the spring constant k of the rope." **[2]**

"State and explain qualitatively how the spring constant of the second sample compares with that of the original sample." **[2]**

Practise on exercise sheet 6.1, then try the real exam questions on past-paper sheet 6.1.

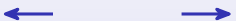
Forces that change shape

A force along the length changes the shape. That change is a **deformation** 形变. Both cases below are measured from the natural length.

Tensile 拉伸

stretches the solid

extension 伸长量 x

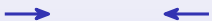


Compressive 压缩

squeezes the solid

compression 压缩,

a negative x



Tensile testing machine

The applied force is the **load** 负载. A spring's own mass is usually **negligible** 可忽略的, so the forces on it are the load and the tension it provides.

Hooke's law

For small extensions the extension is proportional to the load. This is **Hooke's law** 胡克定律.

$$F = kx \qquad k = \frac{F}{x} \qquad [k] = \mathrm{N\,m^{-1}}$$

The **spring constant** 劲度系数 k is the force per unit extension.

> Not "extension per unit force" (that is $1/k$), and not $\frac{1}{2}Fx$ (that is the energy stored).

$2.0\,\mathrm{N}$ stretches a spring $4.0\,\mathrm{cm}$

$4.0\,\mathrm{cm} = 0.040\,\mathrm{m}$

$$k = \frac{F}{x} = \frac{2.0\,\mathrm{N}}{0.040\,\mathrm{m}} = 50\,\mathrm{N\,m^{-1}}$$

Two lengths give k and L_0

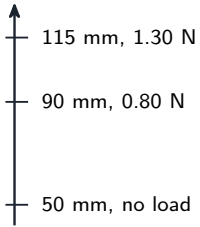
A spring is $90\,\text{mm}$ long under $0.80\,\text{N}$, and $115\,\text{mm}$ under $1.30\,\text{N}$.

The extra $0.50\,\text{N}$ produces an extra $25\,\text{mm} = 0.025\,\text{m}$:

$$k = \frac{\Delta F}{\Delta x} = \frac{0.50\,\text{N}}{0.025\,\text{m}} = 20\,\text{N m}^{-1}$$

At $0.80\,\text{N}$ the extension is

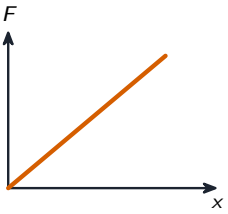
$$x = \frac{F}{k} = \frac{0.80\,\text{N}}{20\,\text{N m}^{-1}} = 0.040\,\text{m} = 40\,\text{mm}.$$



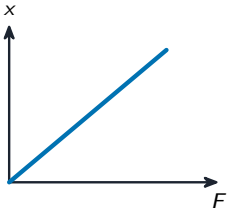
Unstretched length $L_0 = L - x = 90\,\text{mm} - 40\,\text{mm} = 50\,\text{mm}$.

Which gradient is k?

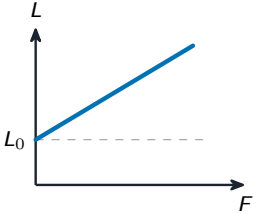
Read the axes before you quote a gradient.



F against x
gradient $= k$



x against F
gradient $= 1/k$

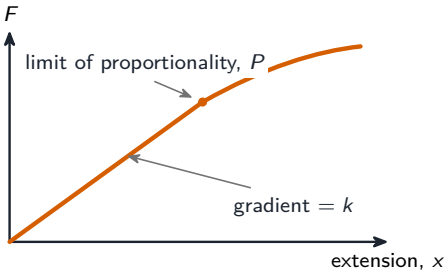


L against F
gradient $= 1/k$

If force is vertical and the horizontal axis is length L rather than extension x, the gradient is still k, because $L = L_0 + x$.

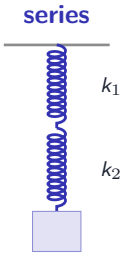
The limit of proportionality

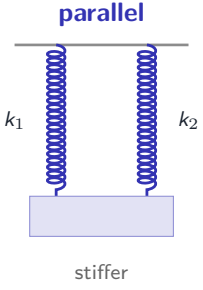
- Hooke's law holds only up to the **limit of proportionality** 比例极限.
- Up to P the F–x line is straight and its gradient is k.
- Past P the line curves. The material may still be **elastic** 弹性 a little further, then it turns **plastic** 塑性.



Springs in series and parallel

- **Series** 串联: the same load passes through both springs and the extensions add. $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$ (softer).
- **Parallel** 并联: identical springs share the load and the extensions are equal. $k = k_1 + k_2$ (stiffer).





A series calculation

Springs 6.0 and $4.0\,\mathrm{N\,cm^{-1}}$, with a load of $80\,\mathrm{N}$, joined end to end

Each spring carries the full $80\,\mathrm{N}$:

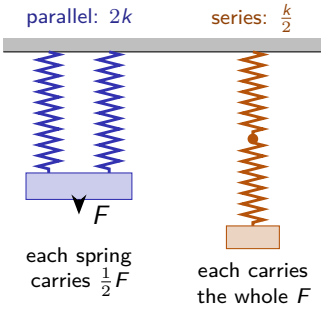
$$x_1 = \frac{F}{k_1} = \frac{80\,\mathrm{N}}{6.0\,\mathrm{N\,cm^{-1}}} = 13.3\,\mathrm{cm}$$

$$x_2 = \frac{F}{k_2} = \frac{80\,\mathrm{N}}{4.0\,\mathrm{N\,cm^{-1}}} = 20\,\mathrm{cm}$$

$$x = x_1 + x_2 = 13.3\,\mathrm{cm} + 20\,\mathrm{cm} = 33\,\mathrm{cm}$$

The same result from $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$, so $k = 2.4\,\mathrm{N\,cm^{-1}}$ and $x = \frac{F}{k} = \frac{80\,\mathrm{N}}{2.4\,\mathrm{N\,cm^{-1}}} = 33\,\mathrm{cm}$.

Sharing a load, and the unit of k



- **Parallel**, identical springs: each carries half the force.
- Energy $\propto F^2$: each stores a quarter, the pair half of one spring's energy.
- Three springs: largest k all in parallel ($3k$), smallest all in series ($k/3$).

$50\,\mathrm{N\,m^{-1}} = 0.50\,\mathrm{N\,cm^{-1}}$: give the unit the question asks for.

2 Stress, strain and the Young modulus

Sheet 6.1.1 – what you need to know

You must be able to:

- define stress, strain and the Young modulus, with units
- calculate E for a wire from its load, extension, length and diameter
- find E from the gradient of a stress–strain or load–extension graph
- describe the wire experiment, and which measurement limits its accuracy

What the exam asks

"Define the Young modulus of a material." **[1]**

"Show that the spring constant k_0 of the wire is given by $k_0 = \dfrac{EA}{L}$." **[2]**

"State and explain the effect, if any, that this has on the strain in the wire." **[2]**

Practise on exercise sheet 6.1.1, then try the real exam questions on past-paper sheet 6.1.1.

Stress and strain

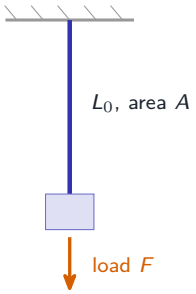
- **Stress** 应力 is the force per unit **cross-sectional area** 横截面积.

$$\sigma = \frac{F}{A} \quad (\mathrm{Pa} = \mathrm{N\,m^{-2}})$$

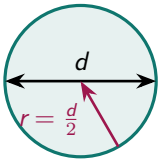
In base units, $\mathrm{Pa} = \mathrm{kg\,m^{-1}\,s^{-2}}$.

- **Strain** 应变 is the extension per unit original length.

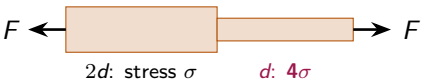
$$\varepsilon = \frac{x}{L_0} \quad \text{(no unit)}$$



Area, and where the stress is larger



$$A = \pi r^2 = \frac{\pi d^2}{4}$$



- Diameter used as the radius: area four times too big. **Halve d first.**
- $1\,\mathrm{mm}^2 = 10^{-6}\,\mathrm{m}^2$.
- Same force all along the wire, so $\sigma \propto 1/d^2$: three times the radius $\Rightarrow$ one ninth of the stress.

18 N on 3.2 mm^2

$$\sigma = \frac{F}{A} = \frac{18\,\mathrm{N}}{3.2 \times 10^{-6}\,\mathrm{m}^2} = 5.6 \times 10^6\,\mathrm{Pa}$$

The Young modulus

The **Young modulus** 杨氏模量 is stress divided by strain, in the Hooke's-law region:

$$E = \frac{\sigma}{\varepsilon} = \frac{F/A}{x/L_0} = \frac{FL_0}{Ax}$$

- Unit: Pa. Metals are about $10^{11}\,\mathrm{Pa}$. Steel is $2.0 \times 10^{11}\,\mathrm{Pa}$.
- E belongs to the material. A thicker wire of the same metal has the same E.
- The spring constant also depends on size: $k = EA/L_0$.

> It stretches less because the area is larger, not because the metal is stiffer.

A steel wire

$L_0 = 2.0\,\mathrm{m}$, $A = 1.5 \times 10^{-7}\,\mathrm{m}^2$, $x = 1.0\,\mathrm{mm}$ under $15\,\mathrm{N}$

The extension must be in metres: $1.0\,\mathrm{mm} = 1.0 \times 10^{-3}\,\mathrm{m}$.

$$E = \frac{FL_0}{Ax} = \frac{(15\,\mathrm{N})(2.0\,\mathrm{m})}{(1.5 \times 10^{-7}\,\mathrm{m}^2)(1.0 \times 10^{-3}\,\mathrm{m})} = 2.0 \times 10^{11}\,\mathrm{Pa}.$$

A thicker steel wire has this same Young modulus.

Size changes k, not E

A "show that $k = EA/L_0$" answer is two lines:

$$k = \frac{F}{x}, \qquad E = \frac{FL_0}{Ax}, \quad \text{so} \quad E = \frac{kL_0}{A} \quad \text{and} \quad k = \frac{EA}{L_0}.$$

- While Hooke's law holds, k stays constant, whatever the force.
- Same metal, twice the diameter: four times the area, so four times k. The Young modulus is unchanged.

A copper wire's spring constant

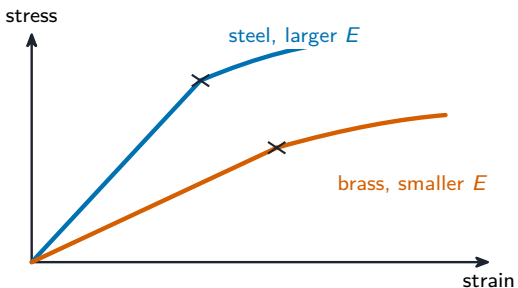
$L_0 = 1.7\,\mathrm{m}$, diameter $0.64\,\mathrm{mm}$, $E = 1.2 \times 10^{11}\,\mathrm{Pa}$

Halve the diameter: $r = 0.32 \times 10^{-3}\,\mathrm{m}$.

$$A = \pi r^2 = \pi(0.32 \times 10^{-3}\,\mathrm{m})^2 = 3.2 \times 10^{-7}\,\mathrm{m}^2$$

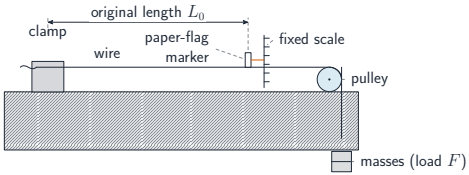
$$k = \frac{EA}{L_0} = \frac{(1.2 \times 10^{11}\,\mathrm{Pa})(3.2 \times 10^{-7}\,\mathrm{m}^2)}{1.7\,\mathrm{m}} = 2.3 \times 10^{4}\,\mathrm{N\,m^{-1}}.$$

Reading E off a graph



- On a stress–strain graph the gradient of the straight part is E.
- The steeper line is the stiffer material.
- Each cross is that metal's limit of proportionality.
- On a force–extension graph the gradient is EA/L_0, so $E = \text{gradient} \times L_0/A$.

Measuring E for a wire



1. Long thin wire over a **pulley** 滑轮.
2. L_0 with a metre rule, from the clamp to a marker.
3. **Diameter** 直径 d with a **micrometer** 螺旋测微器, at several places. $A = \pi d^2/4$.
4. Add loads. Read the extension from the marker against a fixed scale.
5. Plot F against x. In the straight region, $E =$ gradient $\times\, L_0/A$.

The uncertainty in E

$d = (0.50 \pm 0.01)\,\mathrm{mm}$, $x = (2.5 \pm 0.1)\,\mathrm{mm}$; F and L together $\pm 1\%$

diameter: $\dfrac{\Delta d}{d} = \dfrac{0.01\,\mathrm{mm}}{0.50\,\mathrm{mm}} = 2\%$, doubled because $A \propto d^2$: 4%

extension: $\dfrac{\Delta x}{x} = \dfrac{0.1\,\mathrm{mm}}{2.5\,\mathrm{mm}} = 4\%$

$E = \dfrac{FL}{Ax}$ multiplies and divides, so the percentages add:

$$\frac{\Delta E}{E} = 2\frac{\Delta d}{d} + \frac{\Delta x}{x} + 1\% = 4\% + 4\% + 1\% = 9\%$$

$E = (2.0 \pm 0.2) \times 10^{11}\,\mathrm{Pa}$

The diameter is squared, so its percentage uncertainty counts twice: measure it with a micrometer, in several places.

A long wire gives a larger extension, and so a smaller percentage uncertainty in x.

What earns the experiment marks

Measure

L_0 with a metre rule.

d with a micrometer, in several places.

Load from the masses.

Extension from a marker on a fixed scale.

Graph

Plot F against x.
The gradient is EA/L_0.

Or plot stress against strain. That gradient is E itself.

Precautions

Safety goggles: the wire can snap.

A small load first, to take out kinks.

Read on unloading too, to check it stayed elastic.

Long and thin, so the extension is large enough to measure. Repeating d cuts **random error** 随机误差 and checks the wire is uniform.

3 Elastic and plastic

Sheet 6.2 – what you need to know

You must be able to:

- use elastic deformation, plastic deformation and elastic limit
- find the work done from the area under a force–extension graph
- use $E_{\mathrm{P}} = \frac{1}{2}Fx = \frac{1}{2}kx^2$
- explain why energy is not all recovered after plastic deformation

What the exam asks

"Show that the elastic potential energy in the rope is $75\,\mathrm{kJ}$." **[2]**

"Calculate the elastic potential energy E_{P} of the compressed spring." **[2]**

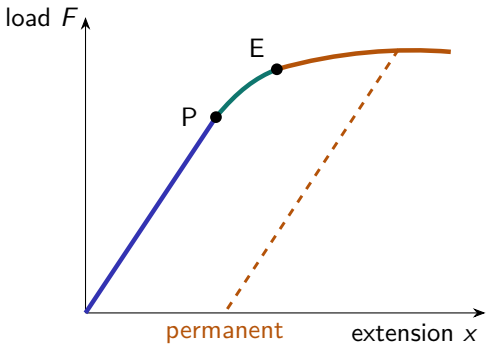
"Explain why the work done in stretching the wire to an extension of $12\,\mathrm{mm}$ is not equal to the energy recovered when the tensile force is removed." **[2]**

Practise on exercise sheet 6.2, then try the real exam questions on past-paper sheet 6.2.

Elastic, then plastic

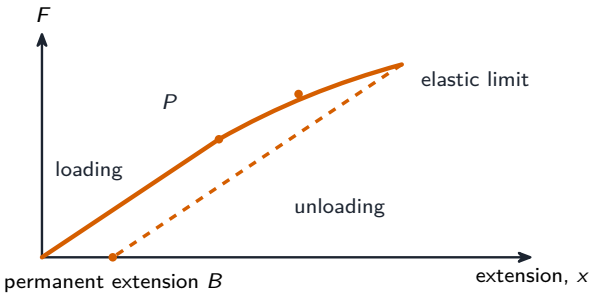
1. **straight, elastic** to the limit of proportionality: Hooke's law holds
2. **curved, still elastic** to the **elastic limit** 弹性极限: unloading still returns
3. **plastic** beyond it: a permanent extension stays

> The end of the straight part is the limit of proportionality; the elastic limit lies a little beyond.



P: limit of proportionality
E: elastic limit

After the elastic limit

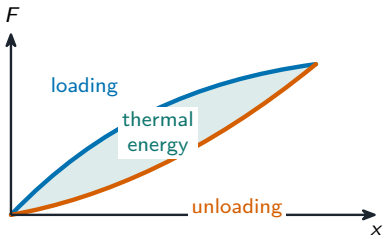


The unloading line is parallel to the first Hooke line, shifted right.

Where it meets the axis, a permanent extension B is left.

> The area between the two lines is energy turned into **thermal energy** 热能.

A rubber band is elastic



A **rubber band** 橡皮筋 returns to its original length, so the deformation is elastic.

The two curves are different. The area between them is **elastic hysteresis** 弹性滞后: energy dissipated as thermal energy.

> Energy recovered on unloading is the area under the lower curve.

A wire taken past its elastic limit does not do this. Part of the deformation is plastic, a permanent extension remains, and part of the work done became thermal energy in the wire.

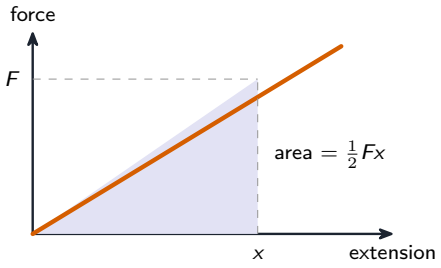
Energy stored is an area

Work done stretching a material is the area under the force–extension graph.

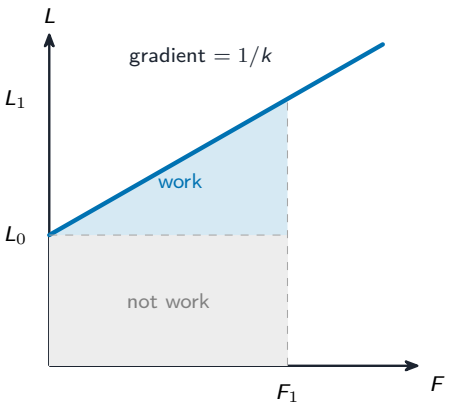
The force grows as the material stretches, so the work is the sum of $F\Delta x$. Within the limit of proportionality the average force is $\frac{1}{2}F$:

$$E_{\mathrm{P}} = \tfrac{1}{2}Fx = \tfrac{1}{2}kx^2 = \frac{F^2}{2k}.$$

E_{P} here is energy, not the Young modulus.



Work on a length–force graph



- The intercept on the length axis is L_0.
- The gradient is $1/k$, not k.
- Work done is the triangle between the line and the level $L = L_0$.
- The rectangle below L_0 is the unstretched length times the force. It is not work.

One joule in a spring

$k = 50\,\mathrm{N\,m^{-1}}$, stretched by $0.20\,\mathrm{m}$, within the limit of proportionality

$$E_\mathrm{P} = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(50\,\mathrm{N\,m^{-1}})(0.20\,\mathrm{m})^2 = 1.0\,\mathrm{J}.$$

This **elastic potential energy** 弹性势能 is stored because the spring was stretched or compressed inside its limit of proportionality.

Between two extensions

A wire with $k = 2.0 \times 10^4\,\mathrm{N\,m^{-1}}$ is already extended by $2.0\,\mathrm{mm}$. The work to reach $3.0\,\mathrm{mm}$ is the difference of two stored energies:

$$\begin{aligned} W &= \tfrac{1}{2}k(x_2^2 - x_1^2) \\ &= \tfrac{1}{2}(2.0 \times 10^4\,\mathrm{N\,m^{-1}}) \\ &\quad \times \left((3.0 \times 10^{-3}\,\mathrm{m})^2 - (2.0 \times 10^{-3}\,\mathrm{m})^2\right) \\ &= 0.050\,\mathrm{J} \end{aligned}$$

Not $\frac{1}{2}k(\Delta x)^2$. That would be a new stretch starting from zero.

- $E_\mathrm{P} \propto x^2$. Double the extension and the energy is four times larger.
- For $k = 400\,\mathrm{N\,m^{-1}}$ and $0.32\,\mathrm{J}$:

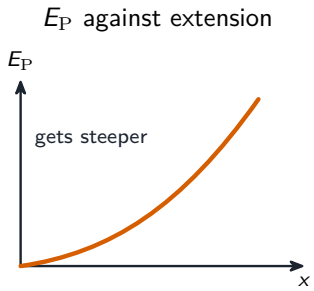
$$\begin{aligned} x &= \sqrt{\frac{2E_\mathrm{P}}{k}} \\ &= \sqrt{\frac{2(0.32\,\mathrm{J})}{400\,\mathrm{N\,m^{-1}}}} = 0.040\,\mathrm{m} \end{aligned}$$

- Two wires end to end feel the same tension. Add $\frac{1}{2}Fx$ for each wire's own extension.

When the line is not straight

Past the limit of proportionality, or for a rubber band, the graph is not a triangle.

- Find the area by counting grid squares (a part square counts as a half) or by **trapezia** 梯形.
- One square is worth the force step times the extension step.
- The area under the whole curve, up to breaking, is still the work done.



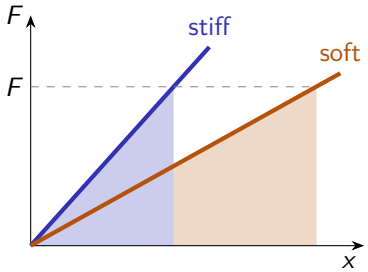
A curve through the origin, not a straight line.

Comparing stored energy

$$E_{\mathrm{P}} = \tfrac{1}{2}Fx = \tfrac{1}{2}kx^2$$

the area under the F–x line

- same force, softer spring: larger x, more energy
- same extension, stiffer spring: larger F, more energy
- same wire size and load: smaller Young modulus stores more



same F: the soft spring's triangle is bigger

Released onto a mass, the elastic potential energy becomes **kinetic energy** 动能: $\frac{1}{2}kx^2 = \frac{1}{2}mv^2$. Add mgh if the mass also rises.

4 Recap

Definitions to memorise

One answer only, in this wording.

Hooke's law	the extension is directly proportional to the applied force, provided the limit of proportionality is not exceeded
spring constant	the force per unit extension
limit of proportionality	the point beyond which the extension is no longer proportional to the applied force
elastic limit	the maximum force (or extension) for which the material returns to its original length when the force is removed
elastic deformation	the material returns to its original length when the force is removed

Five more definitions

plastic deformation	the material does not return to its original length when the force is removed; a permanent extension remains
stress	the force per unit cross-sectional area
strain	the extension per unit original length
Young modulus	the ratio of stress to strain, within the limit of proportionality
elastic potential energy	the energy stored in an object because it has been stretched or compressed

Exam tips

- **Hooke's law ($F = kx$)** holds only up to the limit of proportionality. That condition is part of the statement.
- **Stress** $= F/A$**, strain** $= x/L_0$**, Young modulus = stress/strain** (the gradient of the straight part of a stress–strain graph). The unit is the pascal.
- Energy stored is the **area under the force–extension graph**, and $\frac{1}{2}Fx$ only while the line is straight.
- **Elastic** means it returns to its original length. **Plastic** means a permanent extension remains.

Mistakes that cost the mark

✗	✓
✗ diameter as radius; area in mm^2	✓ halve d; convert to m^2
✗ whole area under a length–force graph	✓ only the triangle above L_0 is **work**
✗ stored energy $\propto$ extension	✓ energy $\propto x^2$
✗ Hooke's law with no limit	✓ state the **limit of proportionality**
✗ limit of proportionality = elastic limit	✓ straight line ends **before** the elastic limit
✗ thicker wire, larger Young modulus	✓ larger k; E is the **material's**

Topic 6 key takeaways

1 Hooke's law

$F = kx$ only up to the limit of proportionality

2 Young modulus

$E = \sigma/\varepsilon$, a property of the material

3 Elastic / plastic

plastic leaves a permanent extension

4 Stored energy

area under F–x; $\frac{1}{2}kx^2$ while straight

Check yourself

1. What is the unit of the spring constant k?
2. A graph plots length against force. What is its gradient?
3. A wire's diameter doubles. What happens to E, and what happens to k?
4. How much energy is stored when $k = 200\,\mathrm{N\,m^{-1}}$ and $x = 0.10\,\mathrm{m}$?

Answers

1. $\mathrm{N\,m^{-1}}$ **2.** $1/k$ **3.** E unchanged, k four times larger **4.** $1.0\,\mathrm{J}$