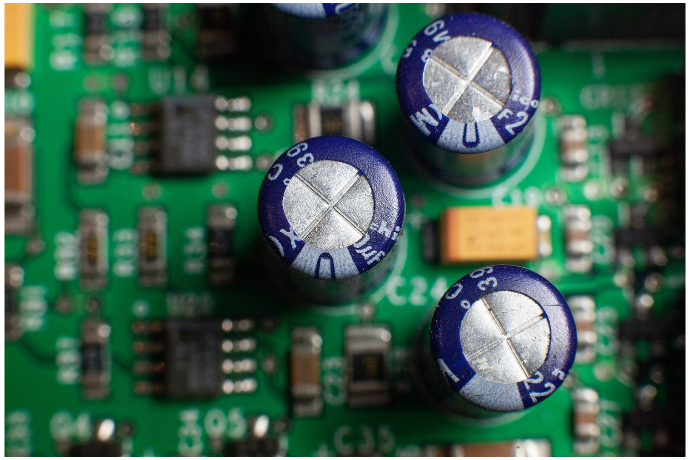
Capacitance

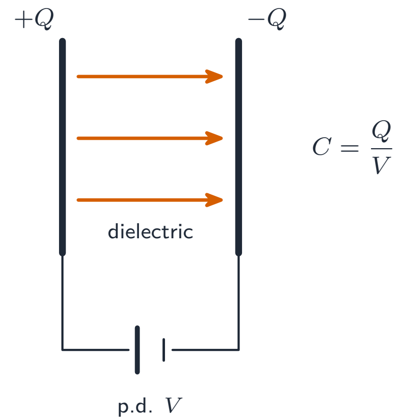
A-Level Physics

Capacitance



Capacitors store electric charge in a circuit.

A **capacitor** 电容器 stores charge. The simplest one is two parallel **conductor** 导体 plates with an **insulator** 绝缘体 (a **dielectric** 电介质, or just **vacuum** 真空 / air) between them. Connected to a battery, charge $+Q$ builds up on one plate and $-Q$ on the other, with a **potential difference** 电势差 V across the gap.



A parallel-plate capacitor: equal and opposite charges on two plates with a p.d. V across the gap

How the plates become charged. Connect the plates to a battery and electrons flow, through the battery, from the plate joined to the positive **terminal** 接线端 to the plate joined to the negative terminal. One plate is left with charge $+Q$ and the other gains $-Q$: the charges are always **equal and opposite**, so the capacitor as a whole stays neutral. The flow stops when the p.d. between the plates equals the e.m.f. of the battery. Nothing crosses the gap; the insulator is what keeps the charge separated.

The **capacitance** 电容 C of any capacitor (or any isolated conductor) is

$$C = \frac{Q}{V}.$$

This applies to:

- an **isolated sphere** holding charge Q at potential V (zero at infinity). For radius r, $V = Q/(4\pi\varepsilon_0 r)$, so $C = 4\pi\varepsilon_0 r$.
- a **parallel-plate capacitor**: charges $\pm Q$ on the plates, p.d. V between them.

Unit: **farad** 法拉 (F) $=$ C V^{-1}. A farad is huge, so real capacitors run from pF to mF.

Capacitance is **constant** for a given capacitor (set by its size and dielectric). Doubling the charge doubles the voltage, so $C = Q/V$ stays the same.

The two-mark definitions. For a **parallel-plate capacitor**: *the charge on one plate per unit potential difference between the plates.* For an **isolated sphere** (or any isolated conductor): *the charge on the conductor per unit potential.* The examiner looks for "charge per unit p.d." (or "per unit potential") and, for the plates, that the charge is the charge on **one** plate and the p.d. is that **between the plates**. "The charge stored" is accepted; "the total charge on both plates" is not, because that total is zero.

What sets the capacitance. For parallel plates the capacitance is proportional to the plate **area** and inversely proportional to the **separation** x ($C \propto 1/x$: halve the gap and the capacitance doubles), and it rises when a dielectric replaces the air. A question that states "C is inversely proportional to x" and then moves the plates apart at constant charge is asking for $V = Q/C \propto x$: the p.d. rises in proportion. At constant p.d. (still connected to the supply) it is the charge that falls instead.

Worked example. A 100 μF capacitor is charged to 12 V. Find the charge stored.

$$Q = CV = (100 \times 10^{-6})(12) = 1.2 \times 10^{-3}\ \text{C}\ (= 1.2\ \text{mC}).$$

Worked example. An isolated metal sphere of radius 15 cm is charged to a potential of 9.0×10^3 V. Find its capacitance and its charge. It is then discharged to earth through a 120 MΩ resistor: find the time constant of the discharge.

$$C = 4\pi\varepsilon_0 r = \frac{0.15}{8.99 \times 10^9} = 1.7 \times 10^{-11}\ \text{F} = 17\ \text{pF}, \qquad Q = CV = (1.67 \times 10^{-11})(9.0 \times 10^3) = 1.5 \times 10^{-7}\ \text{C}.$$

The time constant is $RC = (1.2 \times 10^8)(1.67 \times 10^{-11}) = 2.0 \times 10^{-3}$ s: even through an enormous resistance the sphere loses its charge in milliseconds, because its capacitance is so small. Notice that $4\pi\varepsilon_0 = 1/(8.99 \times 10^9)$, so $C = r/(8.99 \times 10^9)$ with r in metres.

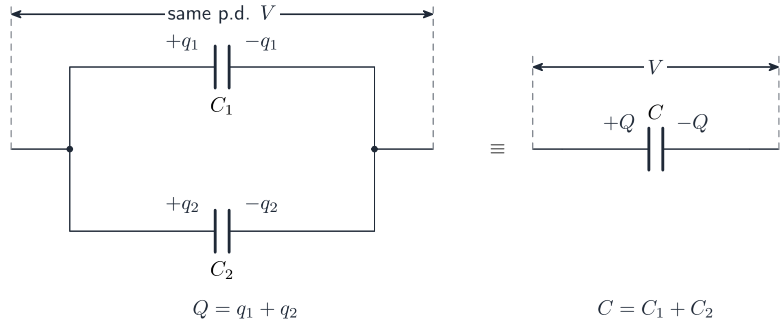
Real capacitors range from large electrolytic cans (high capacitance) down to tiny film and ceramic types – from pF up to mF

Combining capacitors

Capacitors in parallel share the same p.d. V. The total charge is the sum:

$$Q_{\text{total}} = C_1V + C_2V + \ldots, \qquad \text{so} \qquad C_{\text{parallel}} = C_1 + C_2 + \ldots$$

A parallel combination has **larger** capacitance than any one capacitor.

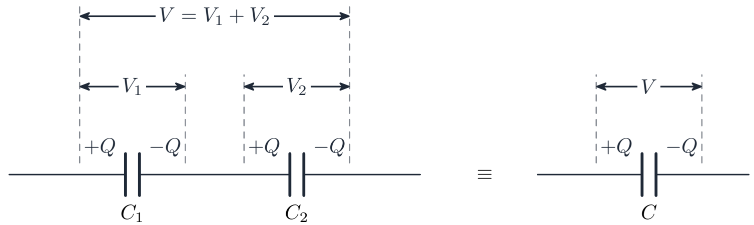


Capacitors in parallel share the same p.d.; the charges add

Capacitors in series carry the **same charge** Q. The total p.d. is the sum:

$$V_{\text{total}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \ldots, \qquad \text{so} \qquad \frac{1}{C_{\text{series}}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots$$

A series combination has **smaller** capacitance than any one capacitor.



Capacitors in series carry the same charge; the p.d.s add

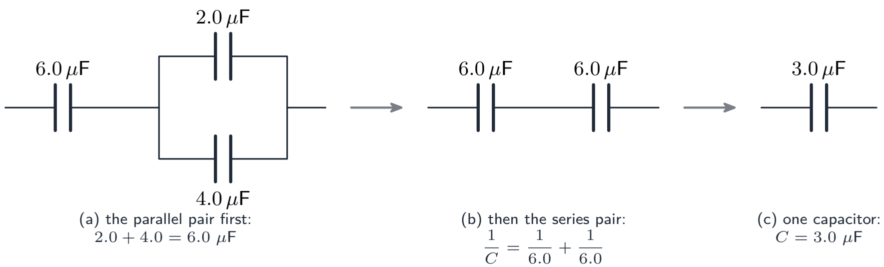
Note: these rules are the **opposite** of those for resistors (resistors sum in series; capacitors sum in parallel), because $C = Q/V$ has V on the bottom while $R = V/I$ has I on the bottom.

Deriving the two rules

"Derive an expression for the combined capacitance" is a **show that** question, and the mark scheme wants the physics stated, not just the algebra.

- **Series.** The capacitors carry the **same charge** Q: the plate joined to the positive terminal loses electrons to the far plate of the next capacitor, so every plate in the chain holds $\pm Q$. The p.d.s **add**: $V = V_1 + V_2$. With $V_1 = Q/C_1$ and $V_2 = Q/C_2$, and the combination defined by $V = Q/C$, dividing through by Q gives $1/C = 1/C_1 + 1/C_2$.
- **Parallel.** The capacitors have the **same p.d.** V, because each is connected directly across the supply. The charges **add**: $Q = Q_1 + Q_2 = C_1V + C_2V$, and $Q = CV$ for the combination, so $C = C_1 + C_2$.

Two capacitors in series always give **less** than the smaller one; two equal capacitors C give $C/2$ in series and $2C$ in parallel.



A network is reduced one step at a time: combine the parallel pair first, then the series pair

Worked example. A 6.0 μF capacitor is in series with a parallel pair of 2.0 μF and 4.0 μF. Find the total capacitance, and the charge on each capacitor when 12 V is applied across the network.

Parallel pair: $2.0 + 4.0 = 6.0$ μF. In series with 6.0 μF: $1/C = 1/6.0 + 1/6.0$, so $C = 3.0$ μF. Total charge $Q = CV = (3.0 \times 10^{-6})(12) = 3.6 \times 10^{-5}$ C: this is the charge on the 6.0 μF capacitor (series, same charge), and it is shared by the parallel pair in the ratio of their capacitances (same p.d., 6.0 V across each): the 2.0 μF holds 1.2×10^{-5} C and the 4.0 μF holds 2.4×10^{-5} C. Check: they add to 3.6×10^{-5} C.

Worked example. Capacitors X and Y, each of capacitance C, are connected in series across a supply of voltage V; capacitors P and Q, each C, are connected in parallel across an identical supply. Compare the charge drawn from each supply and the energy stored in each arrangement.

Series: $C_s = C/2$, charge $Q_s = CV/2$, energy $\frac{1}{2}C_sV^2 = CV^2/4$. Parallel: $C_p = 2C$, charge $2CV$, energy CV^2. The parallel arrangement draws **four times** the charge and stores **four times** the energy, because at the same supply voltage the energy is $\frac{1}{2}CV^2$ and the capacitance is four times larger. Asked which arrangement stores more energy from a given supply, the answer is always parallel.

Energy stored in a capacitor

Charging a capacitor from 0 to Q needs work, because each extra bit of charge is pushed against the p.d. already there. When the charge is q, the p.d. is $V(q) = q/C$, so adding a small charge dq needs work $V\,dq$. The total work is

$$W = \int_0^Q \frac{q}{C}\,dq = \frac{Q^2}{2C}.$$

Using $V = Q/C$, this is the **energy 能量** stored:

$$W = \tfrac{1}{2}QV = \tfrac{1}{2}CV^2 = \frac{Q^2}{2C}.$$

Worked example. Find the energy stored in a 100 μF capacitor charged to 12 V.

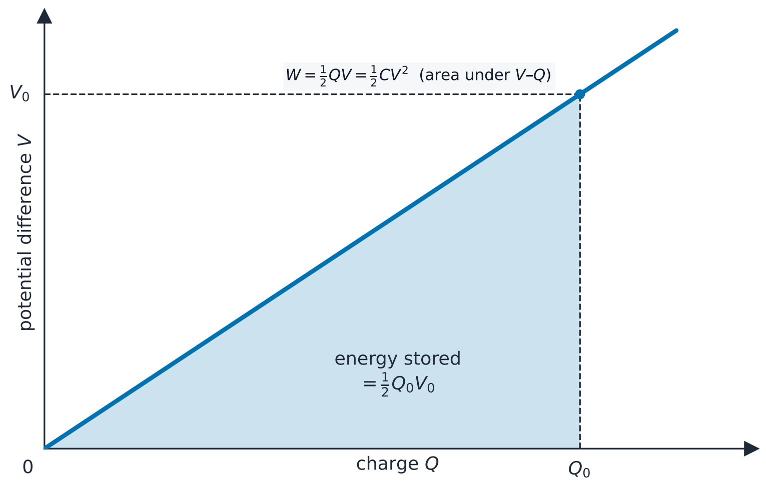
$$W = \tfrac{1}{2}CV^2 = \tfrac{1}{2}(100 \times 10^{-6})(12)^2 = 7.2 \times 10^{-3}\ \text{J}\ (= 7.2\ \text{mJ}).$$

Worked example. The graph of charge Q against p.d. V for a capacitor is a straight line through the origin passing through (10 V, 1.2×10^{-3} C). Find the capacitance and the energy stored at 10 V, and then the extra energy stored when the p.d. is raised to 12 V.

The gradient of Q against V is $C = 1.2 \times 10^{-3}/10 = 1.2 \times 10^{-4}$ F $= 120$ μF. Energy is the area under the line: $\frac{1}{2}QV = \frac{1}{2}(1.2 \times 10^{-3})(10) = 6.0 \times 10^{-3}$ J. At 12 V: $\frac{1}{2}CV^2 = \frac{1}{2}(1.2 \times 10^{-4})(12)^2 = 8.6 \times 10^{-3}$ J, so the extra energy is 2.6×10^{-3} J. Do not find the extra energy from the extra charge as $\frac{1}{2}(\Delta Q)V$: energy is not proportional to charge, so subtract the two areas.

Reading the Q–V graph

A plot of V against Q is a straight line through the origin with gradient $1/C$. The energy stored is the **area** under the line up to a given charge Q, which is the triangle $\frac{1}{2}QV$. The factor $\frac{1}{2}$ is there because the average p.d. during charging is $V/2$ (it grows from zero to V), not V.

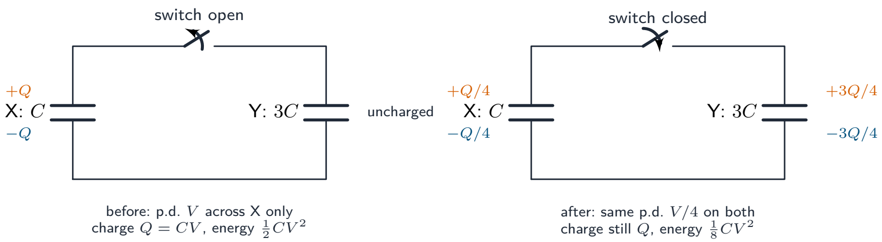


The energy stored is the area under the potential–charge line (the triangle $\frac{1}{2}QV$)

Why charging is "half efficient"

Connect a capacitor C to an ideal battery of e.m.f. V through a wire. The capacitor stores $\frac{1}{2}CV^2$, but the battery supplies charge $Q = CV$ at e.m.f. V, giving out $QV = CV^2$. The other half is lost as heat in the wire — whatever the wire's resistance.

Sharing charge between capacitors



Connecting a charged capacitor to an uncharged one: charge is conserved, the p.d. equalises, and energy is lost

A fully charged capacitor X (capacitance C, p.d. V, charge $Q = CV$) is disconnected from its supply and connected across an uncharged capacitor Y of capacitance $3C$. Charge flows until the two p.d.s are equal. Two principles settle everything:

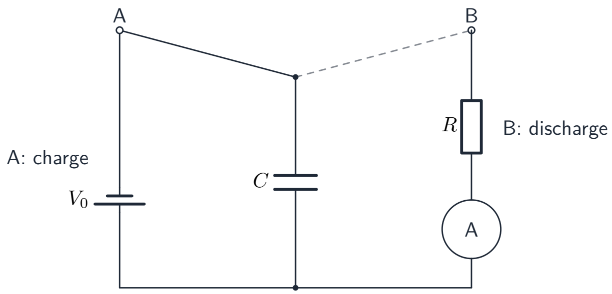
1. **Charge is conserved.** The total charge is still Q, now spread over a parallel pair of total capacitance $4C$, so the common p.d. is $V' = Q/(4C) = V/4$. X keeps $Q/4$ and Y takes $3Q/4$ (charge in the ratio of the capacitances, since the p.d. is the same).
2. **Energy is not conserved.** Before: $\frac{1}{2}CV^2$. After: $\frac{1}{2}(4C)(V/4)^2 = \frac{1}{8}CV^2$. Three-quarters of the energy has gone: it is **dissipated** 耗散 as heat in the connecting wires (and as a spark or electromagnetic radiation) while the charge moves, however small the resistance.

Worked example. A 220 μF capacitor charged to 9.0 V is connected across an uncharged 440 μF capacitor. Find the final p.d. and the energy lost.

$Q = CV = (220 \times 10^{-6})(9.0) = 1.98 \times 10^{-3}$ C; total capacitance 660 μF, so $V' = 1.98 \times 10^{-3}/(660 \times 10^{-6}) = 3.0$ V. Energy before: $\frac{1}{2}(220 \times 10^{-6})(9.0)^2 = 8.9 \times 10^{-3}$ J; after: $\frac{1}{2}(660 \times 10^{-6})(3.0)^2 = 3.0 \times 10^{-3}$ J; lost: 5.9×10^{-3} J, two-thirds of the original. A capacitor connected to another of capacitance kC always loses the fraction $k/(k+1)$.

Capacitor discharging through a resistor

A capacitor C charged to V_0 is connected through a switch to a **resistor** 电阻器 of **resistance** 电阻 R. When the switch closes at $t = 0$, the capacitor discharges.



A capacitor charges through switch A, then discharges through the resistor via switch B

Setting up the equation

By **Kirchhoff's second law** 基尔霍夫第二定律 around the loop, $V_C = V_R$. Using $V_C = Q/C$, $V_R = IR$ and $I = -dQ/dt$:

$$\frac{Q}{C} = -R\frac{dQ}{dt}.$$

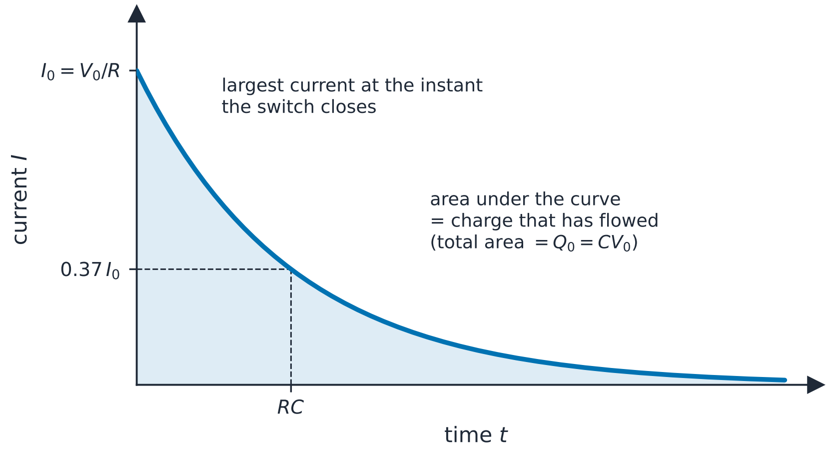
This is solved by an **exponential decay** 指数衰减 with time constant RC.

Discharge equations

Charge Q, p.d. V and current I all decay exponentially with the same time constant:

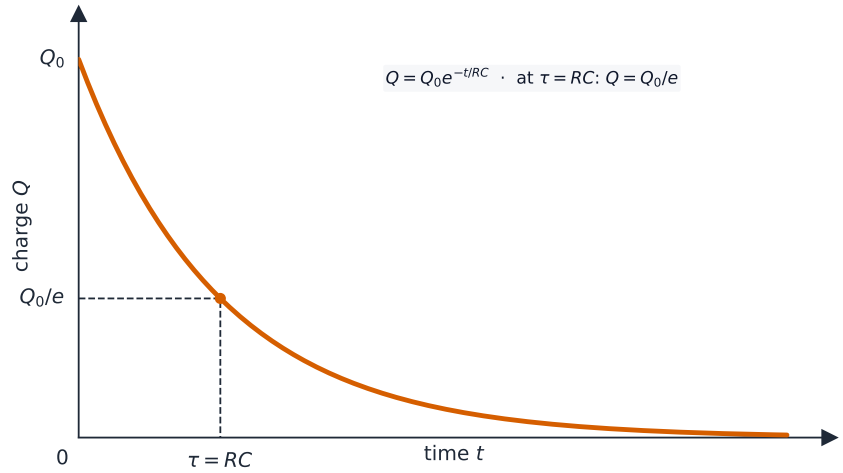
$$Q = Q_0 e^{-t/(RC)}, \qquad V = V_0 e^{-t/(RC)}, \qquad I = I_0 e^{-t/(RC)},$$

with $I_0 = V_0/R$.



The discharge current starts at V_0/R and decays with the same time constant; the area under the curve is the charge that has flowed

Reading the three graphs. All three curves have the same shape and the same time constant, so one measurement of τ from any of them gives RC. The **current** is largest the instant the switch closes, $I_0 = V_0/R$, because the full p.d. V_0 is then across the resistor; it falls as the p.d. falls. The **area** under the I–t graph is the charge that has flowed, so the total area is $Q_0 = CV_0$, and the charge that flows in the first time constant is $Q_0(1 - e^{-1}) = 0.63\,Q_0$. The **gradient** of the Q–t graph is $-I$: steepest at the start, flattening as the current dies away. (During **charging** through a resistor the current has the same decaying shape, while V and Q rise as $V_0(1 - e^{-t/RC})$, the mirror image of the discharge curve.)



Charge decays exponentially during discharge, falling to Q_0/e after one time constant RC

Time constant

$$\tau = RC$$

is the **time constant** 时间常数 (in seconds: $\Omega \cdot \mathrm{F} = \mathrm{s}$). It is the time for a decaying quantity to fall to $1/e \approx 0.37$ (about 37%) of its starting value. After 2τ it is at about 13.5%; after 5τ, below 1%.

Worked example. A 100 μF capacitor charged to 12 V is discharged through a 47 kΩ resistor. Find the time constant and the voltage after one time constant.

$$\tau = RC = (47 \times 10^3)(100 \times 10^{-6}) = 4.7 \text{ s}.$$

After one time constant the voltage falls to $1/e$ of its start: $V = 12 \times 0.37 \approx 4.4$ V.

Worked example. A 470 μF capacitor is charged to 24 V and then discharged through a 5.6 kΩ resistor. Find (a) the initial charge and energy, (b) the time constant and the initial current, (c) the p.d. after 4.0 s, (d) the time for the p.d. to fall to 6.0 V.

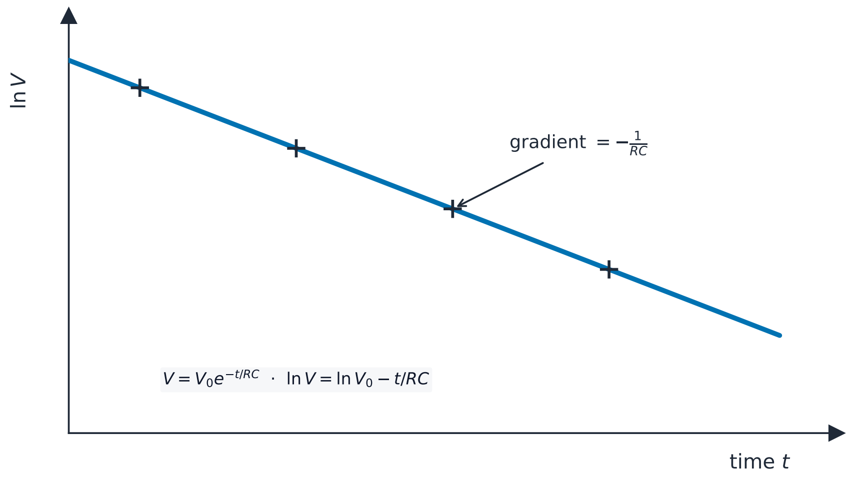
(a) $Q_0 = CV_0 = (470 \times 10^{-6})(24) = 1.1 \times 10^{-2}$ C; $W = \frac{1}{2}CV_0^2 = \frac{1}{2}(470 \times 10^{-6})(24)^2 = 0.135$ J. (b) $\tau = RC = (5.6 \times 10^3)(470 \times 10^{-6}) = 2.6$ s; $I_0 = V_0/R = 24/5600 = 4.3$ mA. (c) $V = V_0 e^{-t/RC} = 24\,e^{-4.0/2.63} = 24 \times 0.219 = 5.3$ V. (d) Rearrange with logarithms: $t = -RC\ln(V/V_0) = -2.63\ln(6.0/24) = 2.63 \times 1.386 = 3.6$ s.

Keep the time constant unrounded (2.632 s) inside the calculation and round only the answers. The rearrangement in (d) is the one most often asked and most often botched: $\ln(V/V_0)$ is negative, so t comes out positive.

Worked example. From a discharge graph, the p.d. across a 2200 μF capacitor falls from 6.0 V to 2.2 V in 4.5 s. Find the resistance of the resistor.

$2.2/6.0 = 0.367 \approx 1/e$, so 4.5 s is one time constant: $R = \tau/C = 4.5/(2200 \times 10^{-6}) = 2.0 \times 10^3\ \Omega$. If the fall is not to $1/e$, use $RC = -t/\ln(V/V_0)$; the answer is the same either way.

To find τ from a curve: read the time to fall to $1/e$ of the start. Or take logs: $\ln(V/V_0) = -t/(RC)$, so a plot of $\ln V$ against t is a straight line with gradient $-1/(RC)$.



A graph of $\ln V$ against time is a straight line of gradient $-1/(RC)$

Using the logarithmic graph. In an experiment, plot $\ln V$ (or $\ln I$) on the y-axis against t. The points should lie on a **straight line**: its gradient is $-1/(RC)$, so $RC = -1/\text{gradient}$, and its **intercept** 截距 on the $\ln V$ axis is $\ln V_0$. A straight line is the test that the decay really is exponential; a curve means it is not. Take the gradient from two points far apart on the best-fit line, not from two data points, and quote RC in seconds.

Reading graphs during discharge

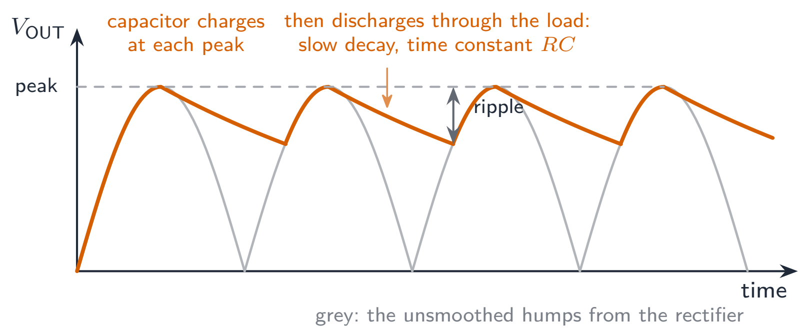
- Q against V_C: since $Q = CV$ always, this is a straight line through the origin with gradient C. Discharge moves the point from (V_0, Q_0) down to $(0, 0)$.
- I against V_C: since $I = V_C/R$, this is a straight line through the origin with gradient $1/R$, so you can find R.

Common exam questions

Given a discharge curve $V(t)$ or $Q(t)$:

- read the start value V_0 or Q_0 at $t = 0$.
- read the time to fall to $V_0/e \rightarrow$ time constant $\tau = RC$.
- given R, find $C = \tau/R$ (or the other way round).
- predict a later value with the exponential formula.

A capacitor as a smoothing component



A capacitor across the load of a rectifier charges at each peak and discharges through the load in between, turning the humps into a nearly steady output with a small ripple

A **rectifier** 整流器 (Topic 21) turns an alternating voltage into a series of positive humps. A capacitor connected **across the load** smooths them. Near each **peak** 峰值 the diodes conduct and the capacitor charges to the peak voltage; as the supply voltage falls away the diodes stop conducting and the capacitor **discharges through the load resistor**, holding the output up until the next hump rises above it and recharges the capacitor. The output therefore falls only a little between peaks; the rise and fall that remains is the **ripple** 纹波.

How much it falls depends on the time constant of the discharge, RC, compared with the time between peaks. A **larger capacitance**, or a **larger load resistance** (a smaller load current), gives a larger RC, a slower decay and a smaller ripple; a heavily loaded supply (small R) has a larger ripple. Asked to sketch the smoothed output, draw it touching each peak, decaying slightly between them, and never falling to zero.

Definitions the examiner accepts

A definition question is marked against fixed wording. Learn these exactly, and give one answer only.

Term	Definition
capacitance (parallel-plate capacitor)	the charge on one plate per unit potential difference between the plates
capacitance (isolated conductor)	the charge on the conductor per unit potential
farad	one coulomb per volt
capacitors in series	the same charge on each; the p.d.s add; $1/C = 1/C_1 + 1/C_2$
capacitors in parallel	the same p.d. across each; the charges add; $C = C_1 + C_2$
energy stored	the work done in charging the capacitor, equal to the area under the potential–charge graph; $W = \frac{1}{2}QV = \frac{1}{2}CV^2$
time constant	the product RC; the time for the charge (or p.d., or current) to fall to $1/e$ of its initial value

Exam tips

- $C = Q/V$ defines capacitance; for an isolated sphere $C = 4\pi\varepsilon_0 r$. Energy stored $W = \frac{1}{2}QV = \frac{1}{2}CV^2 = Q^2/2C$: pick the form that uses what you know, and remember that energy is **not** proportional to charge.
- Combine capacitors the **opposite way to resistors** (parallel add, series reciprocal), and reduce a network one step at a time.
- Charge sharing: **charge is conserved, energy is not**. Find the common p.d. from the total charge and the total capacitance first, then the energies.
- Discharge: $x = x_0 e^{-t/RC}$ for Q, V and I alike; $t = -RC\ln(x/x_0)$; $\tau = RC$ is in seconds ($\Omega \cdot \mathrm{F} = \mathrm{s}$) and is the time to fall to 37%, not to zero.
- Graphs: the initial current is V_0/R; the area under I–t is charge; the gradient of Q–t is $-I$; $\ln V$ against t is a straight line of gradient $-1/(RC)$ and intercept $\ln V_0$.
- Convert μF and kΩ before multiplying: k$\Omega \times \mu$F gives milliseconds, not seconds.

Common mistakes

- Defining capacitance as "the charge stored" or "the charge on both plates". It is the charge on **one** plate per unit p.d. between the plates.
- Adding capacitors in series as if they were resistors, or forgetting to invert the reciprocal sum at the end.
- After charge sharing, using the **supply** voltage in $\frac{1}{2}CV^2$ instead of the new common p.d., or claiming the energy is conserved.
- Drawing the discharge current starting from zero. It starts at its largest value, V_0/R.
- Treating τ as the time to discharge completely; after 5τ about 1% remains.
- Dropping the minus sign in $t = -RC\ln(V/V_0)$ and reporting a negative time.
- Finding the extra energy on raising the p.d. from the extra charge alone: subtract two areas (or two $\frac{1}{2}CV^2$ values).
- Sketching a smoothed output that falls to zero between peaks, or saying a larger load resistance gives a larger ripple.