Oscillations

A-Level Physics

Simple harmonic motion: definition

A pendulum clock keeps time using simple harmonic motion.

A particle moves with **simple harmonic motion** 简谐运动 (SHM) when its **acceleration** 加速度 is:

- **proportional to its displacement** from a fixed **equilibrium** 平衡 point, and
- **directed back towards that point** — opposite in sign to the displacement.

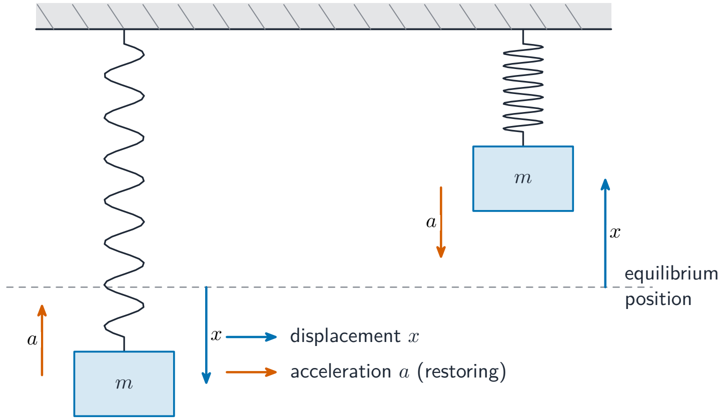
The defining equation is

$$a = -\omega^2 x,$$

where x is the **displacement** 位移 from equilibrium and ω is a positive constant, the **angular frequency** 角频率. The minus sign means "directed back towards equilibrium".

The two-mark definition. *Simple harmonic motion is motion in which the acceleration is proportional to the displacement from a fixed point and is always directed towards that point* (in the opposite direction to the displacement). Both halves are needed. “The significance of the minus sign”: the acceleration is in the **opposite direction** to the displacement — always towards the equilibrium position. “State what is meant by the frequency of the oscillations”: *the number of complete oscillations per unit time.*

Many systems do this near a stable equilibrium: a mass on a **spring** 弹簧, a **pendulum** 单摆 (small swing), a floating block pushed down, the charge on a **capacitor** 电容器 in an LC circuit, atoms in a solid.



Acceleration always points back towards equilibrium, opposite to the displacement

Key terms

- **displacement** x — distance from equilibrium at a moment (a vector along the line of motion).
- **amplitude** 振幅 x_0 — the largest displacement from equilibrium. Always positive.
- **period** 周期 T — the time for one full oscillation.
- **frequency** 频率 f — the number of oscillations per second; $f = 1/T$. Unit: Hz.
- **angular frequency** ω — $\omega = 2\pi/T = 2\pi f$. Unit: rad s^{-1}.
- **phase difference** 相位差 — the fraction of a cycle (in radians) by which one oscillation leads or lags another. A quarter-cycle apart is a phase difference of $\pi/2$.

So $T = 2\pi/\omega$ and $f = \omega/(2\pi)$ — given any one of ω, f, T you can find the others.

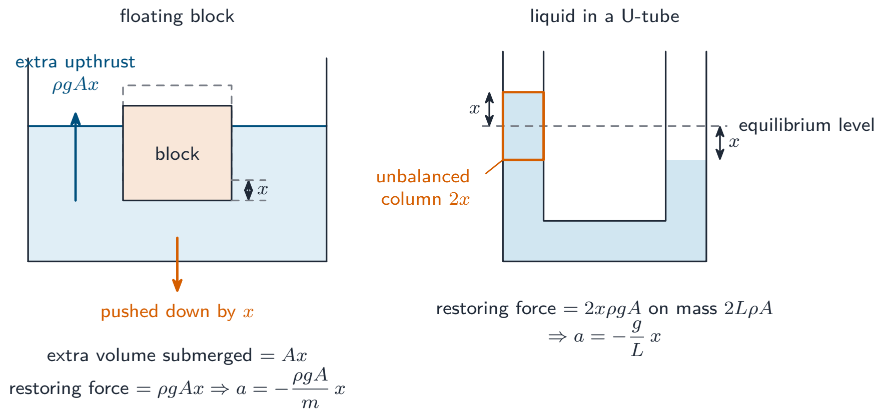
Worked example. A mass on a spring oscillates with SHM of amplitude 0.050 m and frequency 2.5 Hz. Find its maximum acceleration.

The angular frequency is $\omega = 2\pi f = 2\pi \times 2.5 = 15.7$ rad s^{-1}. The acceleration is largest at the extremes, where $|a| = \omega^2 x_0$:

$$a_{\max} = \omega^2 x_0 = 15.7^2 \times 0.050 \approx 12 \text{ m s}^{-2}.$$

Showing that a motion is simple harmonic

The exam's favourite four-mark derivation, and the recipe is always the same: (1) displace the object by x from equilibrium; (2) find the **resultant restoring force**, which comes out as $F = -(\text{constant}) \times x$; (3) apply Newton's second law, $a = F/m = -(\text{constant}/m)\, x$; (4) this has the form $a = -\omega^2 x$, so the motion is simple harmonic with $\omega^2 = \text{constant}/m$, and $T = 2\pi/\omega$. Say in words *why* the force points back towards equilibrium and *why* it is proportional to x — those are the marks, not the algebra.



Two "show that it is SHM" set-ups: find the extra force when the system is displaced by x, and show it is proportional to x and directed back

Worked example. A mass m hangs from a spring of spring constant k. Show that, when displaced and released, it moves with simple harmonic motion, and find the period.

At equilibrium the spring tension balances the weight. Displace the mass a distance x downwards: the tension increases by kx (Hooke's law), so there is a resultant force kx **upwards**, towards equilibrium. Newton's second law: $a = -\dfrac{k}{m}x$. This is $a = -\omega^2 x$ with $\omega^2 = k/m$, so the motion is simple harmonic and $T = 2\pi/\omega = 2\pi\sqrt{m/k}$. (A trolley held between *two* identical springs has both pulling it back, $F = -2kx$, so $\omega^2 = 2k/m$.)

Worked example. A cuboidal block of cross-sectional area A and mass m floats in a liquid of density ρ with its base a depth h below the surface. It is pushed down a small distance x and released. Show that it oscillates with simple harmonic motion and find the period in terms of h and g.

When the block is pushed down by x an extra volume Ax is submerged, so the **upthrust** 浮力 increases by the weight of that extra liquid, $\rho g A x$; this extra force acts **upwards**, towards the equilibrium position. Hence $a = -\dfrac{\rho g A}{m}x$: proportional to x and opposite in direction, so the motion is simple harmonic with $\omega^2 = \rho g A/m$. At equilibrium the upthrust on the submerged volume Ah equals the weight, $\rho g A h = mg$, so $m = \rho A h$ and $\omega^2 = g/h$: $T = 2\pi\sqrt{h/g}$ — the same form as a pendulum of length h.

Worked example. A U-tube of cross-sectional area A contains liquid of density ρ; the column in each arm has length L. The liquid is displaced so that one surface is x above its equilibrium level and the other x below. Show that the liquid oscillates with simple harmonic motion.

The two surfaces now differ in height by $2x$, so a column of liquid of height $2x$ is unbalanced: the restoring force on the liquid is the weight of that column, $2x\rho g A$, directed so as to level the surfaces. The mass being moved is the whole liquid, $2L\rho A$. So $a = -\dfrac{2x\rho g A}{2L\rho A} = -\dfrac{g}{L}x$: simple harmonic, with $\omega^2 = g/L$ and $T = 2\pi\sqrt{L/g}$. Note that ρ and A cancel — the period depends on neither the liquid nor the tube.

Displacement, velocity, acceleration in SHM

If the particle starts at $x = 0$ moving in the positive direction at $t = 0$, then

$$x = x_0 \sin(\omega t).$$

Differentiating once gives the **velocity** 速度:

$$v = x_0\omega\cos(\omega t) = v_0\cos(\omega t),$$

where $v_0 = x_0\omega$ is the **maximum speed** (as the particle passes through equilibrium).

Differentiating again gives the acceleration:

$$a = -x_0\omega^2\sin(\omega t) = -\omega^2 x,$$

which is the SHM defining equation again. (If the particle instead starts at the extreme position $x = x_0$ at $t = 0$, use $x = x_0\cos(\omega t)$. Choose the one that fits the start conditions.)

Velocity in terms of displacement

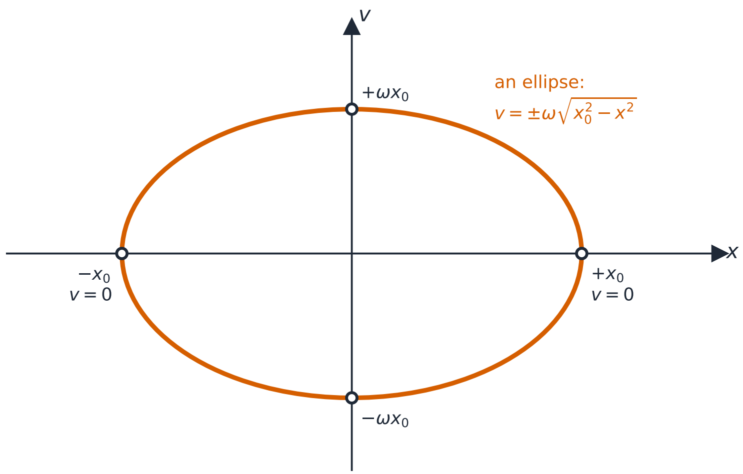
A useful relation that does not use time:

$$v = \pm\omega\sqrt{x_0^2 - x^2}.$$

- at equilibrium ($x = 0$): $v = \pm\omega x_0$ (maximum speed). Both signs, because the particle passes through equilibrium twice each cycle.
- at the extremes ($x = \pm x_0$): $v = 0$ (at rest for an instant).

Worked example. The same oscillation has amplitude $x_0 = 0.050$ m and angular frequency $\omega = 15.7$ rad s^{-1}. Find the speed when the displacement is $x = 0.030$ m.

$$v = \omega\sqrt{x_0^2 - x^2} = 15.7\sqrt{0.050^2 - 0.030^2} = 15.7 \times 0.040 \approx 0.63 \text{ m s}^{-1}.$$



Velocity against displacement is an ellipse: $v = 0$ at the extremes and $v = \pm\omega x_0$ through equilibrium — read x_0 and v_0 from where it cuts the axes, then $\omega = v_0/x_0$

Reading the v–x graph. Its width gives the amplitude ($x = \pm x_0$ where $v = 0$) and its height the maximum speed ($v = \pm\omega x_0$ at $x = 0$), so $\omega = v_0/x_0$ and the period follows. Asked to sketch a against x on the same axes, draw the straight line through the origin with negative gradient $-\omega^2$, reaching $\mp\omega^2 x_0$ at $x = \pm x_0$.

Worked example. The velocity of an oscillating block varies with time as $v = 0.56\sin(5.1t)$, with v in m s^{-1} and t in seconds. Find the amplitude, the period and the maximum acceleration.

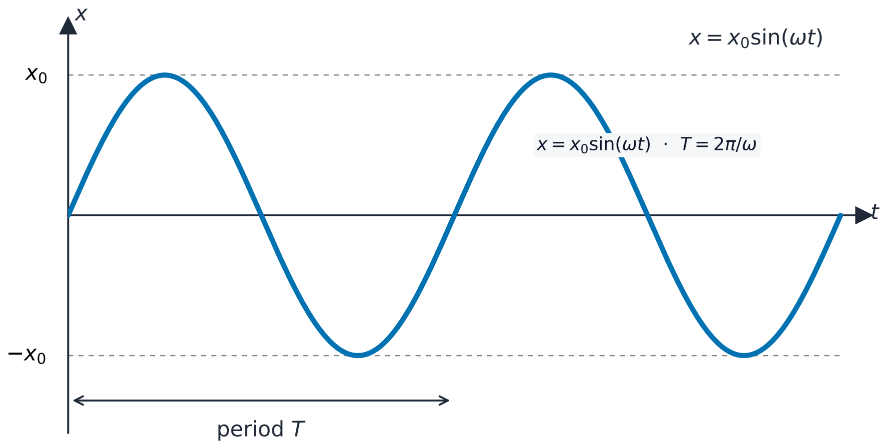
Compare with $v = v_0 \sin\omega t$: $v_0 = 0.56$ m s^{-1} and $\omega = 5.1$ rad s^{-1}. Since $v_0 = \omega x_0$, the amplitude is $x_0 = 0.56/5.1 = 0.11$ m; $T = 2\pi/\omega = 1.2$ s; $a_{\max} = \omega^2 x_0 = \omega v_0 = 5.1 \times 0.56 = 2.9$ m s^{-2}. (Because this v is a *sine*, the block was at an extreme at $t = 0$: its displacement is a cosine.)

SHM as the shadow of circular motion. A ball moving round a circle of radius R at angular speed ω, lit from the side, casts a shadow that moves back and forth with simple harmonic motion of amplitude R and the same ω: the shadow's displacement is $x = R\cos\omega t$ (if it starts at an extreme), its maximum speed is $R\omega$ and its maximum acceleration $R\omega^2$. For a circle of diameter 0.46 m and $\omega = 1.9$ rad s^{-1} the shadow has amplitude 0.23 m, $v_0 = 0.44$ m s^{-1} and $a_{\max} = 0.83$ m s^{-2} — greatest in size at the two ends of its path and always directed back towards the centre.

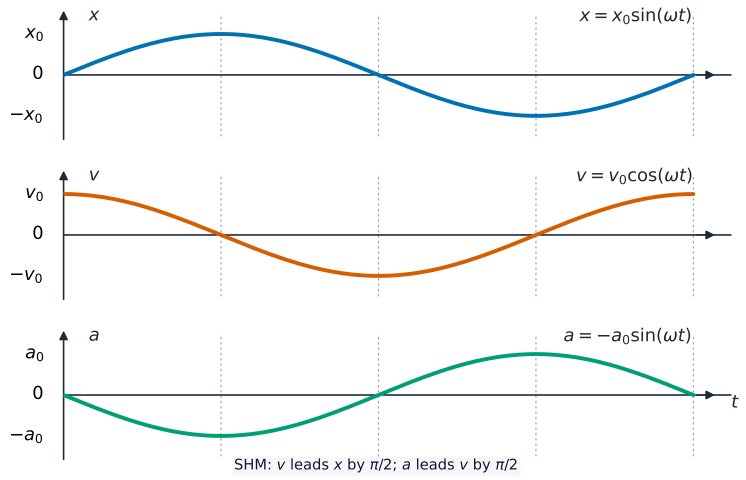
Graphs against time

For $x = x_0 \sin\omega t$:

- x vs t — a sine curve, amplitude x_0, period $T = 2\pi/\omega$.
- v vs t — a cosine curve, **leading** x **by** $\pi/2$, amplitude ωx_0.
- a vs t — a negative sine curve, **out of phase with** x **by** π (180°), amplitude $\omega^2 x_0$.



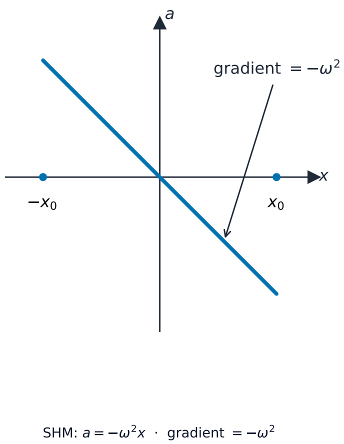
Displacement varies sinusoidally with time in simple harmonic motion



Velocity leads displacement by a quarter cycle; acceleration is exactly out of phase with displacement

Graph of a against x

A straight line through the origin with **negative gradient** $-\omega^2$. So you can read ω from the graph: gradient $= -\omega^2$, so $\omega = \sqrt{|\text{gradient}|}$, then $T = 2\pi/\omega$. This is a common exam pattern.



The acceleration–displacement graph is a straight line through the origin with gradient $-\omega^2$

Worked example. A graph of the height h of an object of mass 36 kg undergoing vertical simple harmonic motion shows h varying between 2 m and 10 m, with one complete oscillation every 4.0 s. Find the amplitude, the angular frequency, the maximum speed, the maximum acceleration and the maximum kinetic energy.

Amplitude $x_0 = (10-2)/2 = 4.0$ m (half the peak-to-peak range; the equilibrium height is 6 m). $\omega = 2\pi/T = 2\pi/4.0 = 1.57$ rad s^{-1}. $v_{\text{max}} = \omega x_0 = 6.3$ m s^{-1}; $a_{\text{max}} = \omega^2 x_0 = 9.9$ m s^{-2}; $E_{\text{K,max}} = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2} \times 36 \times 6.28^2 = 7.1 \times 10^2$ J. Reading the amplitude as the full height range instead of half of it is the commonest slip here.

Worked example. A small object rests on a horizontal platform that oscillates vertically with simple harmonic motion of amplitude 6.0 mm. Find the highest frequency at which the object stays in contact with the platform throughout, and state where in the motion contact is first lost.

The object leaves the platform when the platform's downward acceleration exceeds g — at the **top** of the oscillation, where the acceleration is greatest and directed downwards: the platform then falls away faster than the object can. The limit is $\omega^2 x_0 = g$: $\omega = \sqrt{9.81/0.0060} = 40$ rad s^{-1}, so $f = \omega/2\pi = 6.4$ Hz. Below this the normal contact force at the top is small but positive; above it, zero.

Energy in simple harmonic motion

A simple harmonic oscillator keeps swapping energy between two forms:

- **kinetic energy** 动能 $E_K = \frac{1}{2}mv^2$.
- **potential energy** E_P (elastic for a spring, gravitational for a pendulum).

With no **damping** 阻尼, the **total energy** is constant (this is **conservation of energy** 能量守恒).

Maximum and minimum

- at **equilibrium** ($x = 0$): v is largest, so E_K is largest and E_P is smallest (zero, by choice).
- at the **extremes** ($x = \pm x_0$): $v = 0$, so $E_K = 0$ and E_P is largest.

Total energy

Using $v_{\text{max}} = \omega x_0$:

$$E_{\text{total}} = \tfrac{1}{2}mv_{\text{max}}^2 = \tfrac{1}{2}m\omega^2 x_0^2.$$

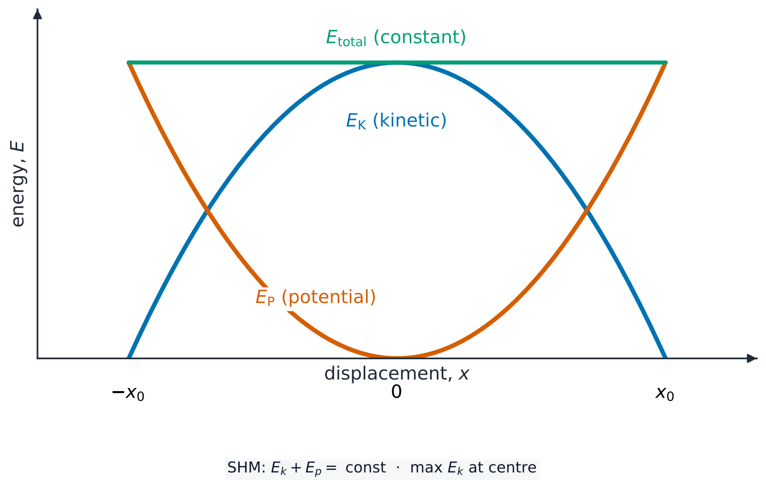
Two key facts: the total energy is proportional to the **square of the amplitude** (doubling x_0 gives four times the energy), and to ω^2.

Energy against displacement

Using $v^2 = \omega^2(x_0^2 - x^2)$:

$$E_K = \tfrac{1}{2}m\omega^2(x_0^2 - x^2), \qquad E_P = \tfrac{1}{2}m\omega^2 x^2.$$

So E_K is a downward parabola (peak at $x = 0$, zero at $x = \pm x_0$) and E_P is an upward parabola (zero at $x = 0$, largest at $x = \pm x_0$). Their sum is constant.



Kinetic and potential energy swap over a cycle while the total energy stays constant

Worked example. A 0.20 kg mass oscillates with amplitude $x_0 = 0.050$ m and angular frequency $\omega = 15.7$ rad s^{-1}. Find the total energy of the oscillation.

The total energy equals the maximum kinetic energy, as the mass passes through $x = 0$ at $v_{max} = \omega x_0$:

$$E = \tfrac{1}{2}m\omega^2 x_0^2 = \tfrac{1}{2}(0.20)(15.7)^2(0.050)^2 \approx 0.062 \text{ J}.$$

This stays constant, swapping between kinetic and potential form twice each cycle.

Describing the interchange without calculation. At an extreme of the motion the oscillator is momentarily at rest: its kinetic energy is zero and its potential energy is a maximum. As it moves towards equilibrium the restoring force does work on it, so

potential energy is converted to kinetic energy; at the equilibrium position the kinetic energy is a maximum and the potential energy a minimum. Beyond it the kinetic energy is converted back to potential energy until the other extreme. The total stays constant, and the exchange happens **twice per oscillation**, so the energy graphs against time have twice the frequency of the displacement graph.

Worked example. A pendulum bob of mass 0.81 kg swings with small oscillations of amplitude 32 mm and period 1.9 s. Find the total energy of the oscillation, and state what happens to it if the amplitude falls to 16 mm.

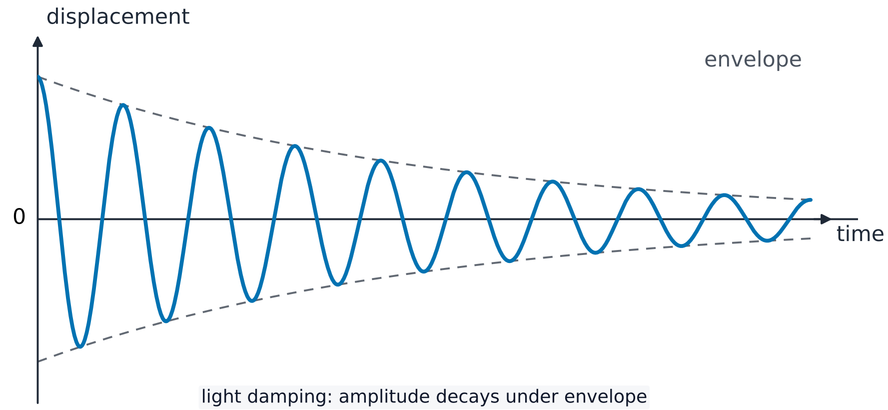
$\omega = 2\pi/1.9 = 3.31$ rad s^{-1}; $E = \tfrac{1}{2}m\omega^2 x_0^2 = \tfrac{1}{2} \times 0.81 \times 3.31^2 \times 0.032^2 = 4.5 \times 10^{-3}$ J. Halving the amplitude quarters the energy ($E \propto x_0^2$), to 1.1×10^{-3} J; the difference has been dissipated as thermal energy by air resistance.

Damped oscillations

A resistive force (**friction** 摩擦力, **drag** 阻力, **air resistance** 空气阻力) causes **damping** — the amplitude shrinks over time as energy is lost as heat. Three named cases:

Light damping

The amplitude **shrinks slowly** over many cycles (a **light damping** 轻阻尼 case). The system still oscillates near its natural frequency, but each cycle is smaller than the last. A car's suspension is light-to-medium damped, so bumps die away but the ride stays smooth.



In light damping the amplitude dies away slowly over many cycles

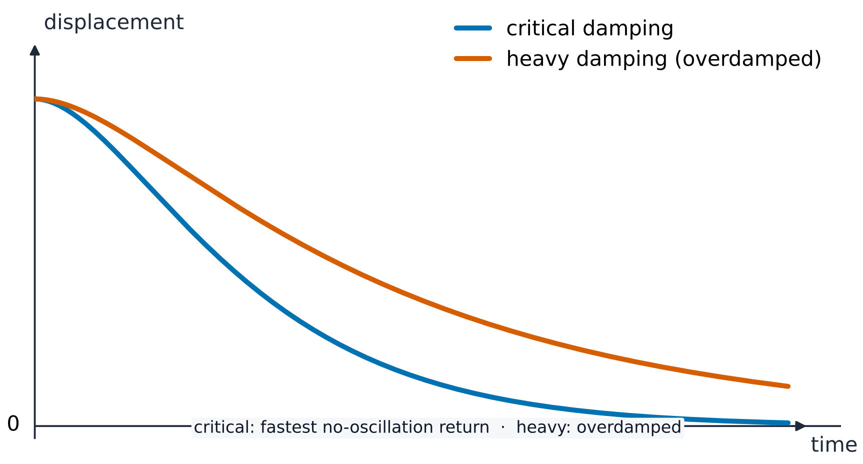
Critical damping

The least damping that brings the system back to equilibrium **without overshooting and without oscillating** — a **critical damping** 临界阻尼 case. It returns in the shortest time. A **galvanometer** 检流计 or analogue **voltmeter** 电压表 is critically damped so the needle settles quickly.

Heavy damping

So much resistance that the system returns slowly, with no oscillation, but more slowly than the critical case — a **heavy damping** 过阻尼 case. A door with a strong closer is heavily damped.

On a displacement–time graph: light damping is a wave whose size dies away smoothly; critical damping returns quickly with no overshoot; heavy damping returns slowly.



Critical damping returns to equilibrium fastest without overshoot; overdamping returns more slowly

As the exam asks it. *Damping* is the decrease in the amplitude of an oscillation caused by a resistive force that dissipates the oscillator's energy. In **light damping** the system oscillates with an amplitude that decreases gradually (exponentially) with time; the period is almost unchanged. In **critical damping** the system returns to equilibrium in the shortest possible time without oscillating. In **heavy damping** it returns to equilibrium slowly, without oscillating. Sketching them: the light-damping curve is a sine wave inside a shrinking envelope; the critical and heavy curves both fall to zero without crossing the axis, the heavy one more slowly.

Worked example. A bar magnet hangs from a spring with one pole inside a coil that is connected through a switch to a resistor. The magnet is set oscillating. Explain why the oscillations die away more quickly when the switch is closed.

The moving magnet changes the magnetic flux linking the coil, so an e.m.f. is induced (Faraday's law); with the switch closed a current flows through the resistor and dissipates energy as heat. By Lenz's law the induced current opposes the motion of the magnet — an extra resistive force. That energy comes from the oscillation, so the amplitude decreases faster: the circuit adds damping. With the switch open there is an e.m.f. but no current, so no extra energy loss.

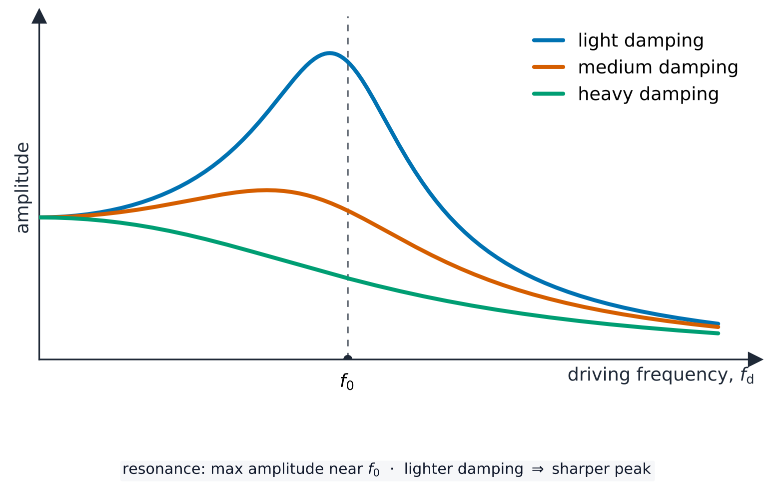
Forced oscillations and resonance

A plucked guitar string vibrates at its resonant frequencies.

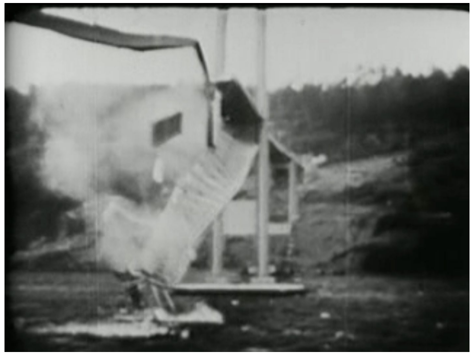
A **forced oscillation** 受迫振动 is driven by an outside periodic force at a frequency f_d chosen by the experimenter. The system then oscillates at this **driving frequency** 驱动频率 f_d, not at its own natural frequency. A plot of amplitude against f_d is a **resonance curve** 共振曲线 with a peak.

Resonance

Resonance 共振 happens when the driving frequency equals the system's **natural frequency** 固有频率 f_0. At resonance the amplitude is **largest** and the energy transfer from the driver is most efficient.



The amplitude of a forced oscillation peaks at resonance, when the driving frequency equals the natural frequency



Resonance can destroy. In 1940 the wind pushed the Tacoma Narrows Bridge close to its natural frequency; with little damping the twisting grew and grew until the deck ripped apart. Engineers now design bridges and buildings so their natural frequencies avoid such driving forces

Examples:

- a swing pushed at the right rate builds up a large amplitude.
- a wine glass broken by a sound at its natural ringing frequency.
- a building shaken by an earthquake whose frequency matches a natural frequency — engineers design buildings so their natural frequencies avoid the main earthquake range.

The peak's shape depends on damping: **lighter damping → a sharper, higher peak**; **heavier damping → a broader, lower peak**, shifted slightly to lower frequency.

The two-mark definition. *Resonance is the condition in which a system is forced to oscillate at its natural frequency, and the amplitude of the oscillation is a maximum* (the transfer of energy from the driver is greatest). Both parts — driving frequency equal to natural frequency, and maximum amplitude — are needed.

Worked example. A ball on a stretched string has a natural frequency of 4.0 Hz. The string is driven by a vibration generator whose frequency is increased slowly from 0 to 10 Hz. Sketch the variation of the amplitude of the ball with the driving frequency, and state the effect of adding damping.

The curve starts at a small, non-zero amplitude at low frequency, rises to a sharp peak at 4.0 Hz, and falls away towards zero at high frequency; the peak is at the natural frequency, and only there is the amplitude a maximum. More damping makes the peak **lower and broader** and shifts it slightly to a lower frequency; the amplitude far from resonance hardly changes.

Definitions the examiner accepts

A definition question is marked against fixed wording. Learn these exactly, and give one answer only.

Term	Definition
simple harmonic motion	motion in which the acceleration is proportional to the displacement from a fixed point and is always directed towards that point
amplitude	the maximum displacement from the equilibrium position
period	the time for one complete oscillation
frequency	the number of complete oscillations per unit time
angular frequency	$\omega = 2\pi f = 2\pi/T$; the rate at which the phase of the oscillation changes
phase difference	the fraction of a cycle, expressed as an angle, by which one oscillation leads or lags another
damping	the reduction in amplitude of an oscillation caused by a resistive force that dissipates its energy
critical damping	the damping that returns a displaced system to equilibrium in the shortest time without oscillation
natural frequency	the frequency at which a system oscillates when displaced and left to oscillate freely

Term	Definition
resonance	the forced oscillation of a system at its natural frequency, at which the amplitude is a maximum

Exam tips

- The **SHM condition is** $a = -\omega^2 x$ (acceleration proportional to displacement, directed back to equilibrium); to *show* a motion is SHM, find the restoring force, show it is proportional to x and directed back, and read off ω^2.
- Learn $x = x_0 \sin\omega t$ (or cos), $v_{max} = \omega x_0$, $a_{max} = \omega^2 x_0$ and $v = \pm\omega\sqrt{x_0^2 - x^2}$; KE and PE interchange with the **total energy constant**, $E = \frac{1}{2}m\omega^2 x_0^2 \propto x_0^2$.
- Velocity is **zero at the extremes** and maximum at the centre; acceleration is greatest at the extremes and zero at the centre.
- The amplitude is **half** the peak-to-peak range of a displacement graph; ω comes from the period, or from the gradient of a–x, or from v_0/x_0.
- **Resonance** occurs when the driving frequency equals the natural frequency; damping lowers and broadens the peak.

Common mistakes

- Defining SHM as "oscillation about an equilibrium position". The mark scheme wants acceleration proportional to displacement *and* directed towards the fixed point.
- Saying the minus sign means the acceleration is negative. It means the acceleration is opposite in direction to the displacement.
- Sketching v against x as a straight line. It is an ellipse; a against x is the straight line.
- Taking the amplitude as the whole range of the displacement. It is half the range.
- Defining resonance as "a large amplitude". It is forced oscillation *at the natural frequency*, giving the *maximum* amplitude.
- Calling critical damping "no damping", or heavy damping "the fastest return". Critical damping is the fastest return without oscillation; heavy damping is slower.