Gravitational fields

A-Level Physics

Gravitational fields

Definition

A **gravitational field** 重力场 is a region where a **mass** 质量 feels a **force** 力 from other masses. The **gravitational field strength** 重力场强度 g at a point is the **gravitational force per unit mass** on a small **test mass** 检验质量 placed there:

$$g = \frac{F}{m}.$$

Unit: N kg^{-1} (the same as m s^{-2} — the acceleration of free fall in the field). g is a **vector** 矢量, pointing the way the force acts — towards the source mass.

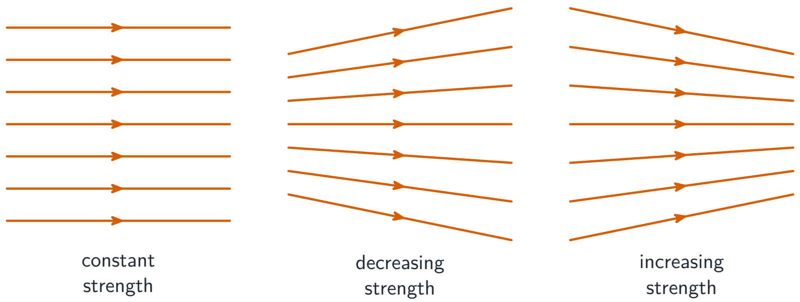
The examiner's wording. A *gravitational field* is a region of space in which a mass experiences a force. *Gravitational field strength* at a point is the gravitational force per unit mass acting on a small test mass placed at the point. The direction of a *field line* at a point is the direction of the force on a (small test) mass placed there. All three are one-mark definitions; "force per unit mass" is the phrase that scores, "the force on 1 kg" is not.

Field lines

A gravitational field is drawn with **field lines** 场线 that point the way the force acts on a test mass:

- around a **point mass** 质点 or a uniform sphere (treated as a point mass from outside), the field lines are **radial** 径向, pointing **inwards**.
- near the Earth's surface over a small area, the field lines are nearly **parallel and equally spaced**, pointing straight down — a **uniform field** 匀强场.

Closer lines mean a stronger field.



Field-line spacing shows the field strength — closer lines mean a stronger field

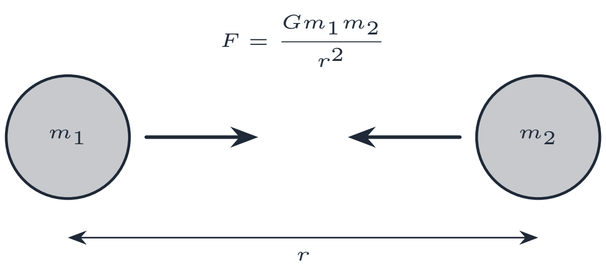
Why g is constant near the surface, in terms of field lines (a two-mark explanation): over a region whose size is small compared with the Earth's radius, the radial field lines are almost parallel and their spacing hardly changes with height, so the field strength — which the spacing represents — is almost the same at the top of the region as at the bottom. Over the whole planet the lines spread out with distance and the field falls.

Worked example. A point P represents a point mass. Draw lines to represent the gravitational field around P, and state what the direction of a line shows.

Straight radial lines, evenly spaced around P, with arrows pointing **inwards** towards P. The direction of a line is the direction of the force on a small test mass placed there — always towards the mass that produces the field, because gravity only attracts.

Newton's law of gravitation

For two point masses m_1, m_2 a distance r apart, the force on each is



Two masses attract along the line joining them

$$F = \frac{Gm_1m_2}{r^2},$$

pulling them together along the line joining them. This is **Newton's law of gravitation** 万有引力定律. The constant $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} is the **universal gravitational constant** 万有引力常量.

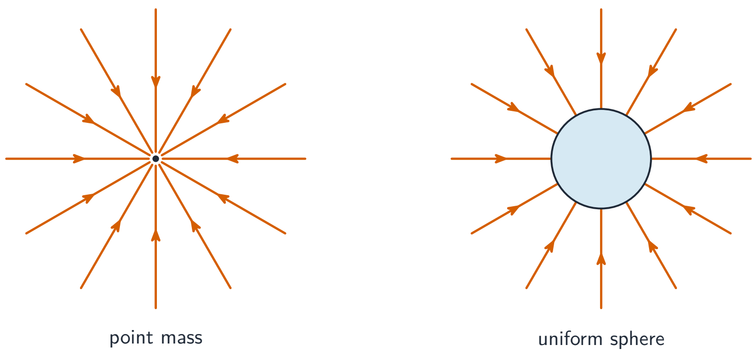
In words, as the mark scheme wants it: *the gravitational force between two point masses is proportional to the product of their masses and inversely proportional to the square of their separation* — both proportionalities, and "point masses". Asked to "state the equation and the meaning of any other symbols", give $F = Gm_1m_2/r^2$ with G the gravitational constant and r the distance between the centres.

Worked example. Two isolated uniform spheres, each of mass 1.0×10^3 kg, have their centres 2.0 m apart. Find the gravitational force between them and comment on its size.

$F = \dfrac{6.67 \times 10^{-11} \times 1.0 \times 10^3 \times 1.0 \times 10^3}{2.0^2} = 1.7 \times 10^{-5}$ N. This is about 10^{-9} of each sphere's weight (9.8×10^3 N): gravity between everyday objects is negligible, and only a planet-sized mass produces a noticeable field. "Isolated" in a question means that no other masses need be considered.

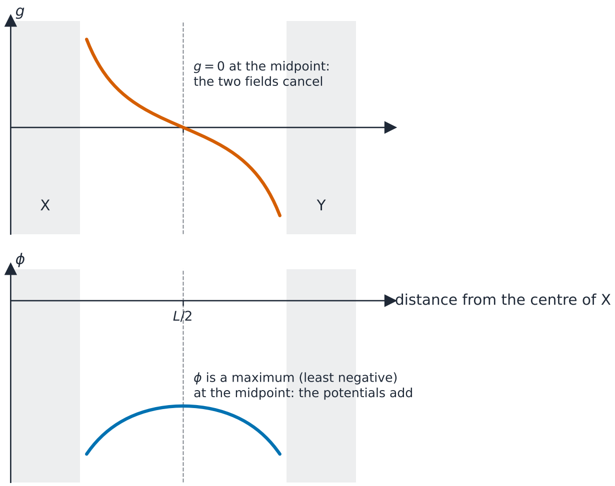
Spheres treated as point masses

For a uniform sphere (such as a planet or star), the field at any point **outside** is the same as that of a point mass equal to the total mass at the centre. So from above the surface, you can treat the Earth as a point mass at its centre. (Points inside a sphere are different, and are not in the syllabus.)



Outside a uniform sphere the field is radial, exactly like a point mass at the centre

Between two masses the fields subtract and the potentials add. Field strength is a vector: on the line joining two spheres the two fields point in opposite directions, so there is a point where they cancel and the resultant field is zero. Potential is a scalar: the two potentials simply add, and both are negative, so the potential is least negative (a maximum) at the point where the field is zero.



Along the line between two equal masses the field strength passes through zero at the midpoint, where the potential has its maximum

Worked example. Two identical isolated uniform spheres X and Y, each of mass M and radius R, have their centres a distance L apart. Point P lies on the line joining the centres. State where on the line the resultant field strength is zero, and find the gravitational potential there.

By symmetry the fields of X and Y are equal and opposite at the **midpoint**, $L/2$ from each centre, so the resultant field is zero there. The potential at P is the sum $\phi = -\dfrac{GM}{L/2} - \dfrac{GM}{L/2} = -\dfrac{4GM}{L}$. A mass released at P would stay there; a mass released anywhere else on the line falls towards the nearer sphere.

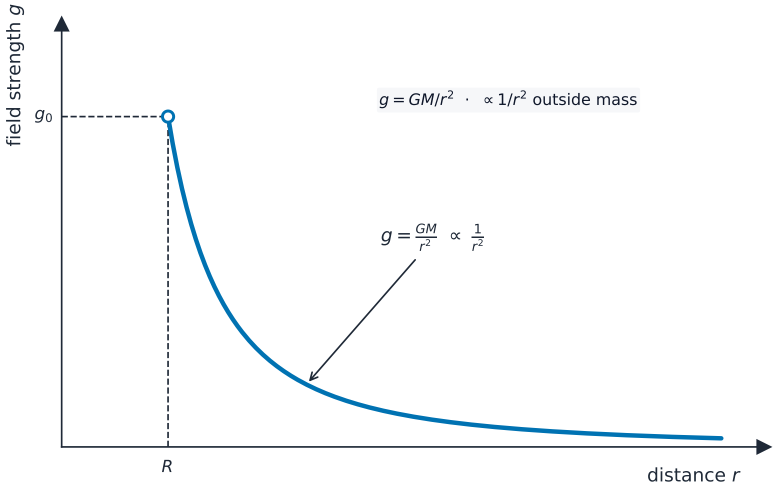
Field strength from a point mass

Put the gravitational force on a test mass m at distance r from a point mass M into $g = F/m$:

$$F = \frac{GMm}{r^2}, \qquad g = \frac{GM}{r^2}.$$

So g falls off as $1/r^2$ as you move away from the source.

The two-mark derivation. The force on a test mass m at distance r from a point mass M is $F = GMm/r^2$ (Newton's law). Field strength is force per unit mass, $g = F/m$, so $g = GM/r^2$. Write both lines: the law and the definition are the two marks. Because $g \propto 1/r^2$, halving the distance makes the field **four** times stronger: at distance $x/2$ the field is $4g$, in the same direction, towards M.



Field strength falls off as $1/r^2$ with distance from a point mass

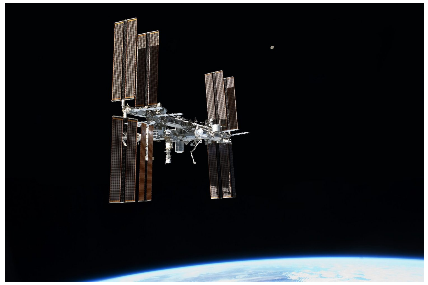
Worked example. Find the gravitational field strength at the Earth's surface. (Earth's mass $M = 6.0 \times 10^{24}$ kg, radius $R = 6.4 \times 10^6$ m, $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.)

$$g = \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11})(6.0 \times 10^{24})}{(6.4 \times 10^6)^2} \approx 9.8 \text{ N kg}^{-1}.$$

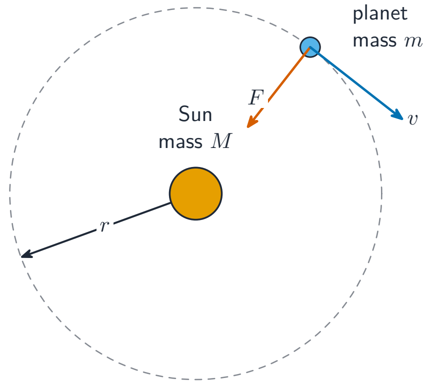
Why g is nearly constant near the Earth's surface

The Earth's radius is $R \approx 6.4 \times 10^6$ m. Rising to height h changes the distance from the centre from R to $R + h$. For $h \ll R$ (any building or mountain), $(R + h)/R \approx 1$, so g barely changes — going from 5 m to 10 m high changes r by about one part in a million. In the laboratory, g is effectively constant.

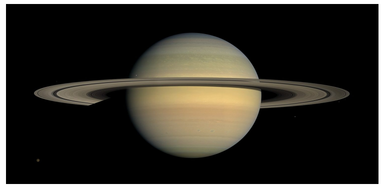
Orbital motion in a gravitational field



The International Space Station orbits Earth, held in its path by gravity.



Gravity provides the centripetal force that keeps a planet in a circular orbit



Saturn, its rings (countless small orbiting pieces) and its moons are all held in orbit by gravity

For a **satellite** 卫星 of mass m in a circular **orbit** 轨道 of radius r around a body of mass M, gravity provides the **centripetal force** 向心力:

$$\frac{GMm}{r^2} = \frac{mv^2}{r}.$$

Cancel m (the orbital speed does not depend on the satellite's mass):

$$v = \sqrt{\frac{GM}{r}}.$$

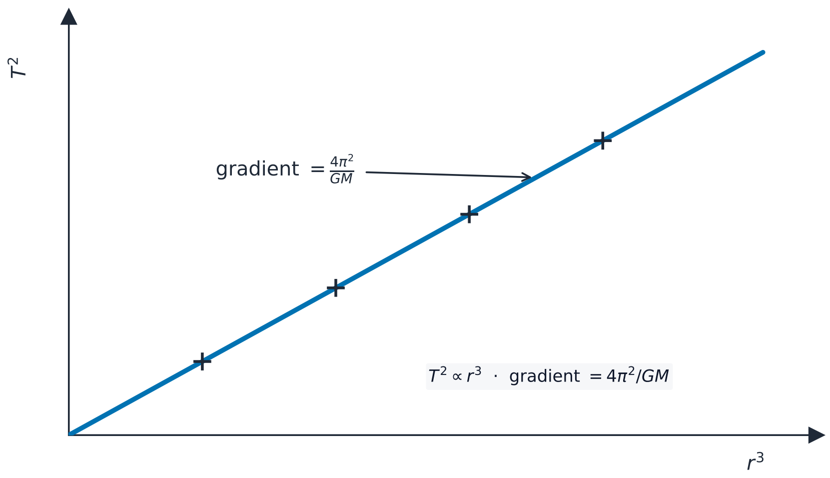
Worked example. A satellite orbits the Earth in a circular orbit of radius $r = 7.0 \times 10^6$ m. Find its orbital speed. (For the Earth, $GM = 4.0 \times 10^{14}$ m^3 s^{-2}.)

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{4.0 \times 10^{14}}{7.0 \times 10^6}} \approx 7.6 \times 10^3 \text{ m s}^{-1}.$$

The **period** 周期 follows from $T = 2\pi r/v$:

$$T = 2\pi\sqrt{\frac{r^3}{GM}}, \quad \text{so} \quad T^2 = \frac{4\pi^2}{GM} \cdot r^3.$$

This is **Kepler's third law** 开普勒第三定律 for circular orbits: $T^2 \propto r^3$. A plot of T^2 against r^3 is a straight line through the origin with gradient $4\pi^2/(GM)$, so orbital data gives the central mass.



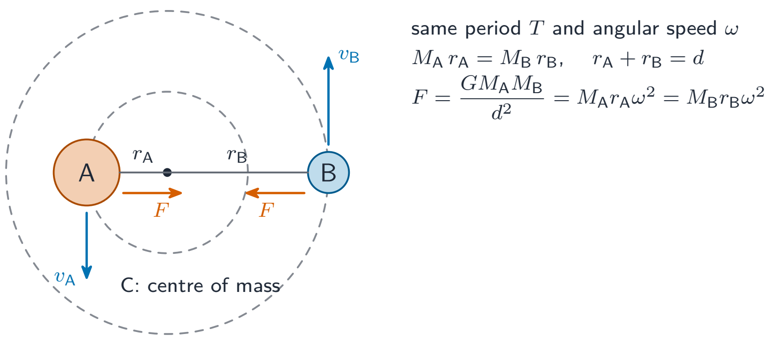
Kepler's third law: $T^2 \propto r^3$, a straight line through the origin

The two-mark derivation, as the exam sets it. For a satellite of mass m in a circular orbit of radius R and period T around a planet of mass M: the gravitational force provides the centripetal force, $\frac{GMm}{R^2} = mR\omega^2$, and $\omega = 2\pi/T$; so $\frac{GM}{R^2} = \frac{4\pi^2 R}{T^2}$, giving $T^2 = \frac{4\pi^2 R^3}{GM}$. For an orbit at height h above a planet of radius R_p the orbital radius is $R_\text{p} + h$, so $T^2 = 4\pi^2(R_\text{p} + h)^3/GM$: a graph of T^2 against $(R_\text{p} + h)^3$ is a straight line through the origin whose gradient gives M.

Worked example. Satellite X, of mass M_s, orbits a planet at a distance $4R$ from its centre; satellite Y, of mass $2M_\text{s}$, orbits at $3R$. Compare their speeds, periods and kinetic energies.

From $v = \sqrt{GM/r}$, $\frac{v_\text{Y}}{v_\text{X}} = \sqrt{\frac{4R}{3R}} = 1.15$: Y moves faster, and the satellite masses do not enter. From $T \propto r^{3/2}$, $\frac{T_\text{Y}}{T_\text{X}} = \left(\frac{3}{4}\right)^{3/2} = 0.65$. Kinetic energy $= \frac{1}{2}mv^2 = \frac{GMm}{2r}$, so $\frac{E_\text{Y}}{E_\text{X}} = \frac{2M_\text{s}/3R}{M_\text{s}/4R} = \frac{8}{3}$. In ratio questions cancel G, M and the satellite mass first; only the radii and masses that differ survive.

Binary stars. Two stars of masses M_A and M_B a distance d apart orbit their common **centre of mass** 质心, always on opposite sides of it, with the **same period**. The centre of mass divides d in the inverse ratio of the masses, $M_\text{A}r_\text{A} = M_\text{B}r_\text{B}$ with $r_\text{A} + r_\text{B} = d$, and the same gravitational force $GM_\text{A}M_\text{B}/d^2$ is the centripetal force on each star.



A binary star: both stars circle the centre of mass with the same period, the heavier one on the smaller circle

Worked example. A binary star consists of star A, mass 4.0×10^{30} kg, and star B, mass 2.0×10^{30} kg, with centres 3.3×10^{11} m apart. Find the distance of A from the centre of mass and the period of the orbit.

$M_A r_A = M_B r_B$ with $r_A + r_B = 3.3 \times 10^{11}$ gives $r_A = \frac{2.0}{6.0} \times 3.3 \times 10^{11} = 1.1 \times 10^{11}$ m (and $r_B = 2.2 \times 10^{11}$ m). For star A: $\frac{GM_A M_B}{d^2} = M_A r_A \omega^2$, so $\omega^2 = \frac{GM_B}{d^2 r_A} = \frac{6.67 \times 10^{-11} \times 2.0 \times 10^{30}}{(3.3 \times 10^{11})^2 \times 1.1 \times 10^{11}} = 1.1 \times 10^{-14}$ s^{-2}, $\omega = 1.06 \times 10^{-7}$ rad s^{-1} and $T = 2\pi/\omega = 6.0 \times 10^7$ s, about 1.9 years. Using d in the force but r_A in the centripetal term is the whole point of the question.

Geostationary orbit

A **geostationary** 地球同步 satellite:

- stays directly above the same point on the Earth (so a fixed dish always points at it),
- has a period of **24 hours** (the same as the Earth's rotation),
- orbits **west to east** (the same way the Earth turns),
- must be directly above the **equator** 赤道.

It must have the same **angular speed** 角速度 as the Earth, in the same direction, in the equatorial plane (or it would drift north–south during the day). From $T = 24$ h and $T^2 = 4\pi^2 r^3/(GM)$, the radius is $r \approx 4.2 \times 10^7$ m (about 3.6×10^7 m above the surface).

Worked example. Calculate the radius of a geostationary orbit around the Earth ($M = 5.98 \times 10^{24}$ kg), and explain why a satellite with the same orbital radius and period may still not be geostationary.

$T = 24 \times 3600 = 8.64 \times 10^4$ s; $r^3 = \frac{GMT^2}{4\pi^2} = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times (8.64 \times 10^4)^2}{4\pi^2} = 7.5 \times 10^{22}$ m^3, so $r = 4.2 \times 10^7$ m — a height of about 3.6×10^7 m above the surface. A satellite with this period is geostationary **only** if its orbit is in the plane of the equator and it travels from west to east; with the same period in a tilted orbit (over the poles, say) it returns to the same point each day but is not always above it. Mars turns once in about 25 hours, so a satellite that stays above one point on Mars has a 25-hour period and the same two other features.

Gravitational potential

Gravitational potential 引力势 ϕ at a point is the **work done per unit mass** in bringing a small test mass from **infinity** 无穷远 to that point:

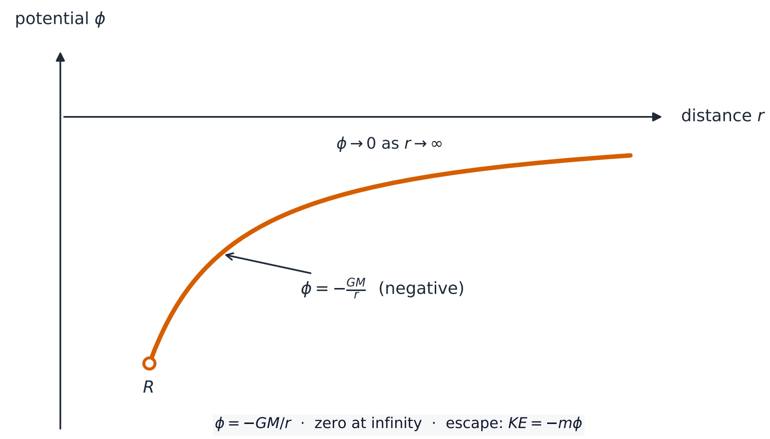
$$\phi = \frac{W}{m}.$$

Unit: J kg^{-1}.

The potential is taken as **zero at infinity**. As the test mass falls in towards the source, gravity does the work for you, so ϕ is **negative** everywhere except at infinity. For a point mass M at distance r:

$$\phi = -\frac{GM}{r}.$$

ϕ is a **scalar** 标量. For several masses, add the potentials.



The gravitational potential well: $\phi = -GM/r$ is negative, rising to zero at infinity

The two-mark definition, and why it is negative. *Gravitational potential at a point is the work done per unit mass in bringing a small test mass from infinity to the point.* Potential is defined as zero at infinity; gravity is attractive, so as the test mass comes in from infinity the field does work *on* it and the mass gives out energy — the work that an external agent must do is negative. Hence the potential at every finite distance is below zero. (Electric potential near a positive charge is positive for the opposite reason: that field repels, so work must be done to bring a positive test charge in.)

Similarities and differences with electric potential. Both are defined as work done per unit mass or per unit positive charge from infinity, both are scalars, both vary as $1/r$ from a point source and both are zero at infinity. The difference: gravitational potential is *always negative*, because gravity only attracts, whereas electric potential can be positive or negative depending on the sign of the charge.

Reading a potential graph. Two graphs appear in questions. On ϕ against r the curve is $-GM/r$: the surface value $\phi_s = -GM/R$ gives the mass of the planet, and the **gradient** at any point gives the field strength there, since $g = -\dfrac{\Delta\phi}{\Delta r}$ — field strength is the negative of the **potential gradient** (this is the "relationship between potential and field strength" a question may ask for). On ϕ against $1/r$ the graph is a straight line through the origin with gradient $-GM$, which is the neater way to extract M.

Worked example. The Moon is an isolated uniform sphere of mass 7.3×10^{22} kg and radius 1.7×10^6 m. Calculate the gravitational potential at its surface, and the minimum speed with which a particle must leave the surface to escape.

$\phi = -\dfrac{GM}{R} = -\dfrac{6.67 \times 10^{-11} \times 7.3 \times 10^{22}}{1.7 \times 10^6} = -2.9 \times 10^6$ J kg^{-1}. To escape, the particle's kinetic energy per unit mass must equal the depth of the potential well: $\frac{1}{2}v^2 = 2.9 \times 10^6$, so $v = \sqrt{2 \times 2.9 \times 10^6} = 2.4 \times 10^3$ m s^{-1}. The minus sign carries the meaning: energy of 2.9 MJ per kilogram must be *supplied* to lift the particle out.

Gravitational potential energy of two point masses

If a test mass m sits where the potential is ϕ, the **gravitational potential energy** 重力势能 of the pair is

$$E_P = m\phi = -\frac{GMm}{r}.$$

Like the potential, E_P is **negative** and reaches zero only at infinite separation. Closer masses have more negative potential energy (more tightly bound).

Link with $\Delta E_P = mg\Delta h$

For small height changes near the surface, r barely changes, so $\Delta E_P \approx mg\Delta h$. For large changes (a satellite moving to a higher orbit) use $-GMm/r$ at each radius and take the difference:

$$\Delta E_P = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad (r_2 > r_1),$$

which is positive (energy must be supplied to raise the satellite).

Worked example. A satellite of mass 1200 kg is moved from a circular orbit of radius 7.0×10^6 m to one of radius 8.0×10^6 m around the Earth ($GM = 4.0 \times 10^{14}$ m^3 s^{-2}). Find the changes in its gravitational potential energy, its kinetic energy and its total energy.

$\Delta E_P = GMm\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right) = 4.0 \times 10^{14} \times 1200 \times \left(\dfrac{1}{7.0 \times 10^6} - \dfrac{1}{8.0 \times 10^6}\right) = +8.6 \times 10^9$ J. In a circular orbit $E_K = \frac{1}{2}mv^2 = \dfrac{GMm}{2r}$, so $\Delta E_K = \dfrac{GMm}{2}\left(\dfrac{1}{r_2} - \dfrac{1}{r_1}\right) = -4.3 \times 10^9$ J: the higher satellite moves more slowly. The total energy $E_K + E_P = -\dfrac{GMm}{2r}$ rises by $+4.3 \times 10^9$ J, which is the energy the rocket motor must supply. A satellite's total energy is negative — it is bound — and it becomes less negative as the orbit widens.

Escape velocity (from conservation of energy)

To escape from radius r to infinity, an object's **kinetic energy** 动能 must equal the size of its gravitational potential energy:

$$\tfrac{1}{2}mv_{esc}^2 = \frac{GMm}{r}, \qquad v_{esc} = \sqrt{\frac{2GM}{r}}.$$

At the Earth's surface, the **escape velocity** 逃逸速度 is ≈ 11 km s^{-1}. It does not depend on the object's mass.

Worked example. Find the escape velocity from the Earth's surface. (For the Earth, $GM = 4.0 \times 10^{14}$ m^3 s^{-2}, $R = 6.4 \times 10^6$ m.)

$$v_{esc} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2(4.0 \times 10^{14})}{6.4 \times 10^6}} \approx 1.1 \times 10^4 \text{ m s}^{-1} \ (= 11 \text{ km s}^{-1}).$$

Worked example. A particle is projected vertically upwards from the surface of an isolated planet of radius R, where the gravitational potential is ϕ_s. Show that it just reaches a distance r from the centre if its launch speed v satisfies $\frac{1}{2}v^2 = -\phi_s\left(1 - \dfrac{R}{r}\right)$, and deduce the escape speed.

Potential varies as $1/r$, so at distance r it is $\phi_s R/r$. When the particle stops, its kinetic energy per unit mass has all become potential energy per unit mass: $\frac{1}{2}v^2 = \phi(r) - \phi_s = \phi_s\dfrac{R}{r} - \phi_s = -\phi_s\left(1 - \dfrac{R}{r}\right)$, which is positive because ϕ_s is negative. Letting $r \to \infty$ gives $\frac{1}{2}v_{esc}^2 = -\phi_s = GM/R$ — the escape-speed result again, now read straight off the potential.

Definitions the examiner accepts

A definition question is marked against fixed wording. Learn these exactly, and give one answer only.

Term	Definition
gravitational field	a region of space in which a mass experiences a force
gravitational field strength	the gravitational force per unit mass acting on a small test mass placed at the point
field line (direction)	the direction of the force on a small test mass placed at that point
Newton's law of gravitation	the gravitational force between two point masses is proportional to the product of their masses and inversely proportional to the square of their separation
gravitational potential	the work done per unit mass in bringing a small test mass from infinity to the point
gravitational potential energy (two point masses)	the work done in bringing the two masses from infinity to their separation, $E_P = -GMm/r$
geostationary orbit	an orbit with a period of 24 hours, from west to east, directly above the equator, so the satellite stays above the same point on the surface
escape speed	the minimum speed at which an object must leave the surface to reach infinity with zero kinetic energy
centre of mass (binary star)	the point about which both stars orbit, dividing their separation in the inverse ratio of their masses

Exam tips

- **Newton's law of gravitation** $F = GMm/r^2$ (inverse-square); field strength $g = GM/r^2$ — and quote the derivation as two lines, the law and the definition of g.
- Distinguish **gravitational potential** ($\phi = -GM/r$, always negative, zero at infinity) from field strength; g is the negative gradient of the ϕ–r graph.
- For an orbit set gravity = centripetal force to get $T^2 \propto r^3$; a geostationary orbit has $T = 24$ h plus two more features: above the equator, west to east.
- Use the **orbital radius** (centre to centre, $R_p + h$), never the height, and for a binary star the separation d in the force but the orbit radius r_A in the centripetal term.

- Energy changes come from $-GMm/r$ at each radius; for a circular orbit $E_K = GMm/2r$ and the total energy is $-GMm/2r$.

Common mistakes

- Defining potential as "force per unit mass" or field strength as "work done per unit mass". Swap them and both marks go.
- Saying the potential is negative "because gravity attracts" and stopping there. The mark needs the work argument: zero at infinity, and the field does the work as the mass comes in.
- Using the height above the surface as r in GM/r^2 or $T^2 \propto r^3$. Add the planet's radius first.
- Writing Kepler's law as $T \propto r$ or $T^2 \propto r^2$. It is $T^2 \propto r^3$: a straight line only on T^2 against r^3.
- Making "geostationary" mean only "24-hour period". Without the equatorial plane and the west-to-east direction it is not geostationary.
- Forgetting that a satellite's kinetic energy *falls* when it is raised to a higher orbit, even though energy had to be supplied.