Motion in a circle

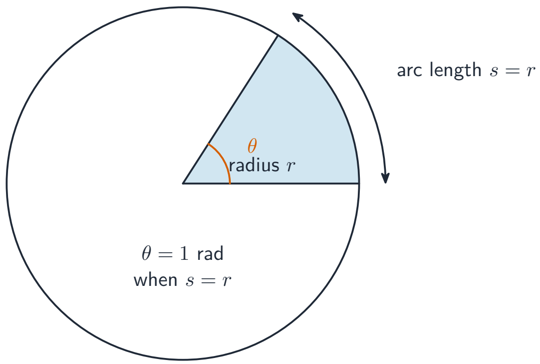
A-Level Physics

Angles in radians

The **radian** 弧度 is the angle made at the centre of a circle by an **arc** 弧 whose length equals the radius. For an arc of length s on a circle of radius r, the angle in radians is

$$\theta = \frac{s}{r}.$$

Radians have no unit (a ratio of lengths). A full circle has $s = 2\pi r$, so $\theta = 2\pi$ rad. A half-circle is π rad; a quarter is $\pi/2$ rad.



One radian is the angle whose arc length equals the radius

To convert: $1 \text{ rad} = 180°/\pi \approx 57.3°$. Set your calculator to **radians** for this topic; "degree" mode will give wrong answers.

The one-mark definition. *The radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.* Give it in words — $\theta = s/r$ on its own is not a definition — and keep $s = r\theta$ ready for turning a distance along the arc into an angle, or back.

Uniform circular motion: angular speed

A spinning fairground ride: every rider turns through the same angle each second.

An object moves in a circle of radius r at constant speed v. Define:

- **angular displacement** 角位移 θ — the angle (in radians) turned through by the radius from a chosen start line.
- **angular speed** 角速度 ω — the rate of change of angular displacement.

For uniform motion ω is constant and

$$\omega = \frac{\theta}{t}.$$

Unit: rad s^{-1}.

What stays constant and what varies. In uniform circular motion the **speed**, the angular speed, the period and the *magnitude* of the acceleration are constant; the **velocity**, the **displacement** from the centre and the **acceleration** all change continuously, because their *direction* changes. Asked to "state two quantities that vary", give two of velocity, acceleration, displacement and momentum — and say it is the direction that varies.

Period and frequency

If the object goes once round (2π rad, one **revolution** 圈) in time T (the **period** 周期), then

$$\omega = \frac{2\pi}{T} = 2\pi f,$$

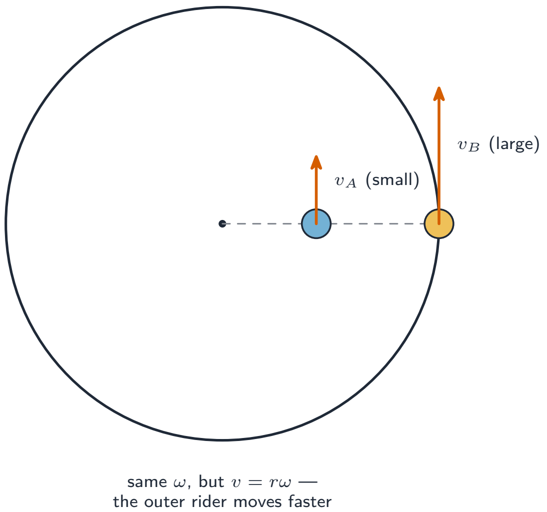
where $f = 1/T$ is the **frequency** 频率 of turning (Hz).

Linear and angular speed

In one period T the object travels a distance $2\pi r$ (the **circumference** 周长) at constant speed, so

$$v = \frac{2\pi r}{T} = r\omega.$$

This links the linear (**tangential** 切向) speed v with the angular speed ω. At a larger radius (for the same angular speed) the linear speed is larger — a child on the edge of a merry-go-round moves faster than one near the centre, even though both go round once in the same time.



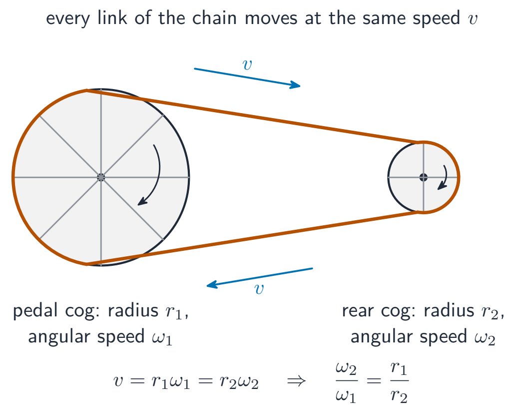
Same angular speed, but the rider at the larger radius has the larger linear speed ($v = r\omega$)

Worked example. A fairground ride of radius 4.0 m completes one turn every 8.0 s. Find its angular speed and the linear speed of a rider on the edge.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8.0} = 0.79 \text{ rad s}^{-1}, \qquad v = r\omega = 4.0 \times 0.79 = 3.1 \text{ m s}^{-1}.$$

Worked example. The minute hand of a clock turns once every hour. A piece of modelling clay on the hand moves a total distance of 0.44 m in 1400 s. Find the angular speed of the hand, the angle the clay turns through in that time, and its distance from the centre of the clock.

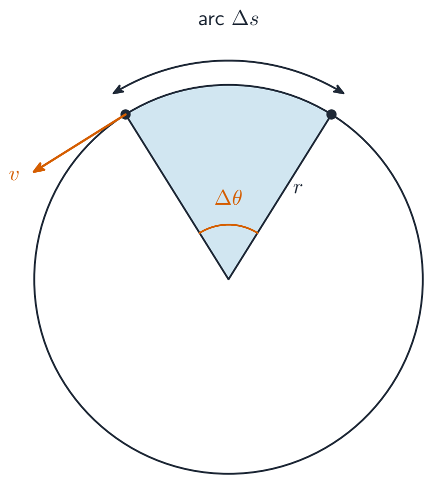
$\omega = 2\pi/T = 2\pi/3600 = 1.75 \times 10^{-3}$ rad s^{-1}. Angular displacement $\theta = \omega t = 1.75 \times 10^{-3} \times 1400 = 2.44$ rad. Distance along the arc $s = r\theta$, so $r = s/\theta = 0.44/2.44 = 0.18$ m. Check: $v = s/t = 3.1 \times 10^{-4}$ m s^{-1} and $r\omega = 0.18 \times 1.75 \times 10^{-3} = 3.1 \times 10^{-4}$ m s^{-1} — the same. A common trap in these questions is a radius measured *from the rim*: a lump 1.2 cm in from the edge of a 9.3 cm disc moves in a circle of radius 8.1 cm.



Two cogs joined by a chain share the chain's speed, not its angular speed: $v = r_1\omega_1 = r_2\omega_2$

Worked example. A bicycle chain passes round a pedal cog of radius 0.095 m and a rear-wheel cog of radius 0.038 m. The pedals turn at 1.2 revolutions per second. Find the speed of the chain and the angular speed of the rear cog. The chain is then moved to a smaller rear cog while the bicycle's speed stays the same; explain, without calculation, what happens to the angular speed of the pedals.

Pedal angular speed $\omega_1 = 2\pi \times 1.2 = 7.5$ rad s^{-1}. Every link of the chain moves at the speed of the rim of the pedal cog: $v = r_1\omega_1 = 0.095 \times 7.5 = 0.72$ m s^{-1}. The rear cog's rim moves at the same speed, so $\omega_2 = v/r_2 = 0.72/0.038 = 19$ rad s^{-1}. With the bicycle's speed unchanged, the rear *wheel* keeps the same angular speed; the smaller cog's rim therefore moves more slowly ($v = r\omega$ with a smaller r), so the chain moves more slowly, and the pedal cog, driven by the same chain, turns at a **lower** angular speed.



As the radius turns through $\Delta\theta$ the object moves an arc Δs at speed v

Centripetal acceleration

An object moving in a circle at constant speed still has a changing **velocity** 速度 — its direction keeps changing, even though its size stays the same. A changing velocity needs an **acceleration** 加速度. This acceleration points **towards the centre** and is the **centripetal acceleration** 向心加速度.

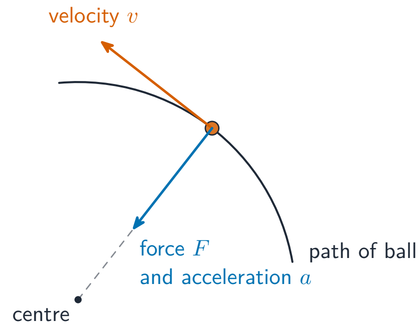
The two-mark description. *In uniform circular motion the speed is constant but the velocity is continuously changing, because its direction changes; the acceleration is constant in magnitude and is always directed towards the centre of the circle, perpendicular to the velocity.* "State what is meant by centripetal acceleration": *the acceleration of an object moving along a circular path, directed towards the centre of the circle.*

Size

$$a = \frac{v^2}{r} = r\omega^2.$$

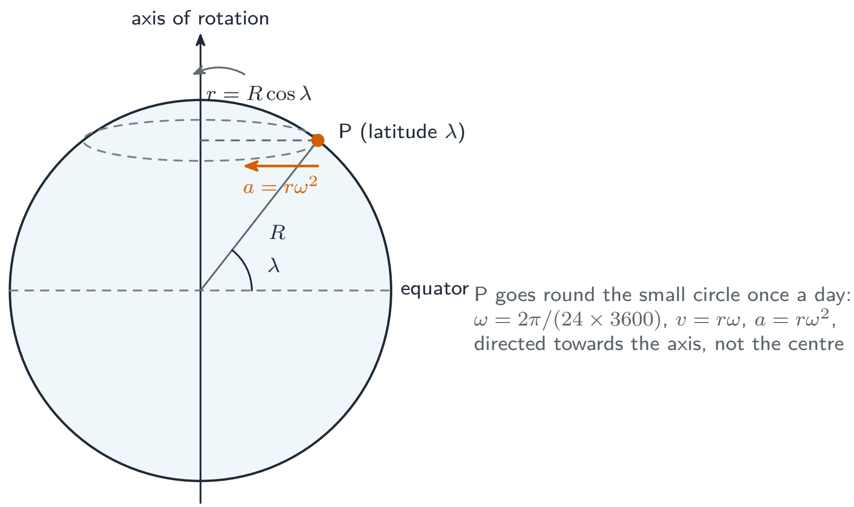
The two forms are equal because $v = r\omega$. Pick the one with the quantities you have.

The centripetal acceleration is **perpendicular** 垂直 to the velocity at every instant — never along the direction of motion. (If part of it were along the motion, the speed would change.) Unit: m s^{-2}.



The velocity points along the tangent; the force and acceleration point to the centre

Why the speed does not change. The centripetal force is perpendicular to the displacement at every instant, so it does **no work** on the object; its kinetic energy, and so its speed, stays constant. Only the direction of the velocity changes — which is exactly what an acceleration perpendicular to the velocity does.



A point on the rotating Earth moves in a circle about the axis, of radius $R\cos\lambda$, not about the Earth's centre

Worked example. The Earth is a sphere of radius 6.37×10^6 m rotating once in 24 hours. Cambridge is at **latitude** 纬度 52.2° north. Find the radius of the circle in which Cambridge moves, its speed, and its centripetal acceleration. A student of mass

58.6 kg stands on bathroom scales there; state and explain the effect of the rotation on the reading.

The circle is about the *axis*, so $r = R\cos\lambda = 6.37\times10^6 \times \cos 52.2° = 3.90\times10^6$ m. $\omega = 2\pi/(24\times3600) = 7.27\times10^{-5}$ rad s^{-1}, so $v = r\omega = 284$ m s^{-1} and $a = r\omega^2 = 3.90\times10^6\times(7.27\times10^{-5})^2 = 2.1\times10^{-2}$ m s^{-2}. The student needs a resultant force $ma = 1.2$ N towards the axis, which can only come from the weight exceeding the upward contact force; so the scales read *less* than the weight $mg = 575$ N — by about one newton (the acceleration has a component $a\cos\lambda = 0.013$ m s^{-2} along the vertical, i.e. 0.7 N of the 1.2 N). Small, but the sign and the reason are the marks.

Centripetal force

A Ferris wheel: a centripetal force toward the centre keeps each car moving in a circle.

By Newton's second law, the resultant **force** 力 on a body in circular motion at constant speed is

$$F = ma = \frac{mv^2}{r} = mr\omega^2.$$

This is the **centripetal force** 向心力. It **always points towards the centre** — perpendicular to the velocity.

The centripetal force is not a new kind of force — it is the **net result** of the real forces acting (tension, gravity, friction, electric attraction, normal contact force, …). In a problem, work out which real force(s) provide it.

Worked example. A 0.20 kg ball on a string is whirled in a horizontal circle of radius 0.50 m at 3.0 m s^{-1}. Find the centripetal force (the tension in the string).

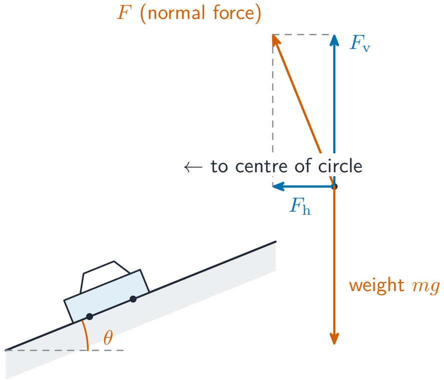
$$F = \frac{mv^2}{r} = \frac{0.20\times3.0^2}{0.50} = 3.6\ \text{N}.$$

Where the centripetal force comes from

- **Ball on a string** in a horizontal circle: the **tension** 张力 in the string.
- **Car turning a flat corner**: the **friction** 摩擦力 between tyres and road ($F = mv^2/r$). If the car goes too fast, friction is not enough and it skids outwards.
- **Banked corner** 倾斜 (no friction): the horizontal part of the **normal contact force** 支持力; $\tan\theta = v^2/(rg)$ for the angle that needs no friction.
- A planet or **satellite** 卫星 in **orbit** 轨道: the **gravitational attraction** 引力, $GMm/r^2 = mv^2/r$.
- **Electron** 电子 in a circular orbit (Bohr-style model): the **electrostatic** 静电 attraction between the electron and the positive **nucleus** 原子核:

$$\frac{kZe^2}{r^2} = \frac{m_e v^2}{r},$$

where $k = 1/(4\pi\varepsilon_0)$ and Z is the nuclear charge. Solve for v to get the orbital speed; then $T = 2\pi r/v$.



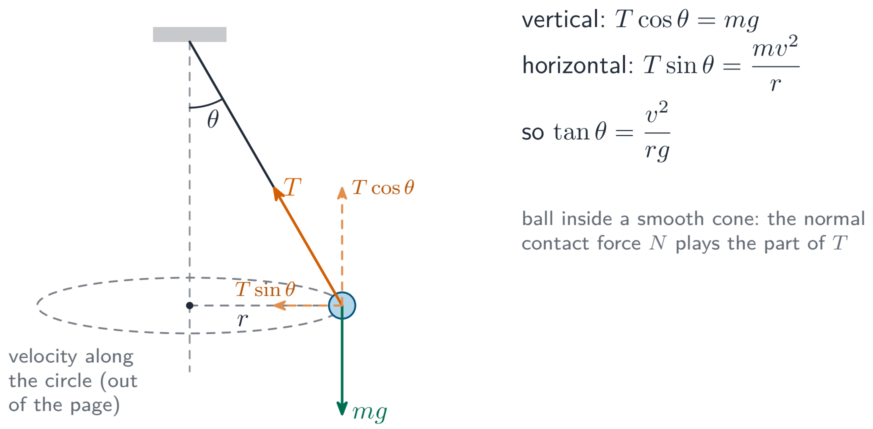
On a banked track the horizontal part of the road's force provides the centripetal force

The tilted force: conical pendulum, cone and swing-ride

A ball on a string swung in a horizontal circle (a **conical pendulum** 圆锥摆), a ball rolling round the inside of a smooth cone, a chair on a fairground swing-ride and a car on a banked track are all the same problem: one force (the tension or the normal contact force) is **tilted** at an angle θ to the vertical. Resolve it. Its **vertical** component balances the weight; its **horizontal** component is the whole centripetal force:

$$F\cos\theta = mg, \qquad F\sin\theta = \frac{mv^2}{r} \qquad \Rightarrow \qquad \tan\theta = \frac{v^2}{rg}.$$

Never add a separate "centripetal force" to the diagram: the two real forces are the weight and the tilted force, and their *resultant* is horizontal, towards the centre.



Resolve the tilted force: vertical component = mg, horizontal component = mv^2/r

Worked example. A steel ball moves in a horizontal circle of radius 0.12 m on the smooth inside surface of a cone whose surface makes 52° with the horizontal. Name the two forces on the ball, state the direction of their resultant, and find the speed of the ball and the period of its motion.

The forces are the **weight** (vertically down) and the **normal contact force** from the cone's surface (perpendicular to the surface, so at 52° to the vertical); their resultant is horizontal, towards the centre of the circle. Vertically $N\cos 52° = mg$; horizontally $N\sin 52° = mv^2/r$. Dividing: $v^2 = rg\tan 52° = 0.12 \times 9.81 \times 1.28 = 1.51$, so $v = 1.2$ m s^{-1}, and $T = 2\pi r/v = 2\pi \times 0.12/1.23 = 0.61$ s. The mass cancels — the speed does not depend on it.

Worked example. A sphere of mass 0.29 kg hangs from a spring of spring constant 40 N m^{-1} and unstretched length 6.0 cm. It is set moving in a horizontal circle so that the spring makes 30° with the vertical. Find the tension, the radius of the circle and the speed.

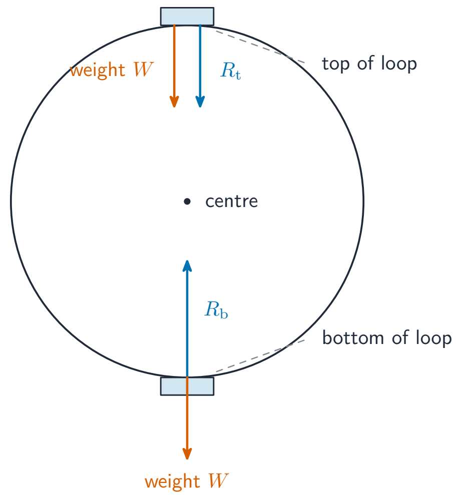
Vertically $T\cos 30° = mg$, so $T = 0.29 \times 9.81/0.866 = 3.3$ N. The extension is $x = T/k = 3.28/40 = 0.082$ m, so the spring's length is $0.060 + 0.082 = 0.142$ m and $r = 0.142\sin 30° = 0.071$ m. Horizontally $T\sin 30° = mv^2/r$: $v^2 = 3.28 \times 0.5 \times$

$0.071/0.29 = 0.40$, $v = 0.63$ m s^{-1}. Hooke's law supplies the length; the circle supplies the rest.

Vertical circles

When the circle is upright, the speed is **not** constant (gravity does work) — but at each instant the **net force towards the centre** still equals mv^2/r:

- at the bottom of a loop: tension up, **weight** 重力 down, so $T - mg = mv^2/r$ — the tension is largest here.
- at the top of a loop: tension and weight both point down (towards the centre), so $T + mg = mv^2/r$ — the tension is smallest. For the slowest speed at the top with the string just tight, set $T = 0$: $mg = mv_{\min}^2/r$, giving $v_{\min} = \sqrt{gr}$.



Forces on a person at the top and bottom of a vertical circle

Worked example. A car goes round a vertical loop of radius 2.0 m. Find the minimum speed at the top for the car to keep contact with the track (take $g = 9.81$ m s^{-2}).

At the slowest speed the track force is zero, so gravity alone provides the centripetal force: $mg = mv_{\min}^2/r$, giving $v_{\min} = \sqrt{gr}$:

$$v_{\min} = \sqrt{9.81 \times 2.0} \approx 4.4 \text{ m s}^{-1}.$$

The constant-speed result ($v = r\omega$, ω constant) holds for horizontal circles, or where the force only bends the path (orbits in gravity, charges in a **magnetic field 磁场**).

Circles in fields

Worked example. A helium atom is modelled as a nucleus of charge $+2e$ with two electrons in the same circular orbit of radius 170 pm, always on opposite sides of the nucleus. Find the resultant electric force on one electron and its speed.

Each electron is attracted by the nucleus, distance r away, and repelled by the other electron, distance $2r$ away, along the same line: $F = \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{2e^2}{r^2} - \dfrac{e^2}{(2r)^2}\right) = \dfrac{e^2}{4\pi\varepsilon_0 r^2} \times 1.75 = \dfrac{2.31 \times 10^{-28}}{(1.7 \times 10^{-10})^2} \times 1.75 = 1.4 \times 10^{-8}$ N. This is the centripetal force: $v = \sqrt{Fr/m} = \sqrt{1.40 \times 10^{-8} \times 1.7 \times 10^{-10}/(9.11 \times 10^{-31})} = 1.6 \times 10^6$ m s^{-1}.

Worked example. A proton (mass 1.67×10^{-27} kg) moving at 2.0×10^6 m s^{-1} enters a uniform magnetic field of flux density 0.50 T at right angles to the field. Find the radius of its path and the time for one revolution.

The magnetic force Bqv is always perpendicular to the velocity, so it is a centripetal force and the speed is constant: $Bqv = mv^2/r$ gives $r = \dfrac{mv}{Bq} = \dfrac{1.67 \times 10^{-27} \times 2.0 \times 10^6}{0.50 \times 1.60 \times 10^{-19}} = 4.2 \times 10^{-2}$ m, and $T = \dfrac{2\pi r}{v} = \dfrac{2\pi m}{Bq} = 1.3 \times 10^{-7}$ s — independent of the speed, which is why a faster proton makes a larger circle in the same time. For a satellite the gravitational force plays the same part: $\dfrac{GMm}{r^2} = mr\omega^2$, and the orbital speed and period follow from r alone.

How to structure a circular-motion answer

1. **Find the radius** r and choose v or ω. Use $v = r\omega$ to switch between them.
2. **Find the centripetal acceleration** with $a = v^2/r$ or $r\omega^2$.
3. **List the real forces** and write Newton's second law in the **radial 径向** direction (towards the centre is positive). Set the net inward force equal to mv^2/r.
4. For **period or frequency**: use $\omega = 2\pi/T$, or $T = 2\pi r/v$.
5. Check the **directions**: centripetal force and acceleration point to the centre; the velocity is along the tangent.

Definitions the examiner accepts

A definition question is marked against fixed wording. Learn these exactly, and give one answer only.

Term	Definition
radian	the angle subtended at the centre of a circle by an arc equal in length to the radius
angular displacement	the angle, in radians, through which the radius has turned from its starting position
angular speed	the angle swept out by the radius per unit time (the rate of change of angular displacement)
period	the time taken for one complete revolution
uniform circular motion	motion in a circle at constant speed, with a velocity that changes continuously in direction and an acceleration of constant magnitude directed towards the centre
centripetal acceleration	the acceleration of an object moving in a circular path, directed towards the centre of the circle and perpendicular to the velocity
centripetal force	the resultant force on an object in circular motion, directed towards the centre, of magnitude mv^2/r

Exam tips

- Work in **radians**; angular speed $\omega = 2\pi/T = v/r$, and arc length $s = r\theta$ turns a distance along the circle into an angle.
- **Centripetal acceleration** $a = v^2/r = \omega^2 r$; the net force acts **towards the centre** — it is provided by tension/gravity/friction/a contact force, not an extra force.
- Always state **what provides the centripetal force** in the situation given, and for a tilted force resolve it: vertical component $= mg$, horizontal component $= mv^2/r$.
- Read the radius carefully: a point measured *from the rim* moves in a smaller circle; a point on the rotating Earth moves in a circle about the **axis**, radius $R\cos\lambda$.
- Two cogs on one chain share the chain's **speed**; two points on one rigid wheel share its **angular speed**.

Common mistakes

- Defining the radian as "$180/\pi$ degrees" or as $\theta = s/r$. The mark needs the arc equal in length to the radius.
- Saying the velocity is constant in uniform circular motion. The *speed* is constant;

the velocity changes direction, which is why there is an acceleration.

- Drawing a “centripetal force” arrow as an extra force alongside the tension or contact force. It is their resultant, not a third force.
- Using degrees in ωt or $v = r\omega$. Every formula in this topic assumes radians.
- Forgetting that the force does no work: a force perpendicular to the motion changes direction, not speed, so kinetic energy is constant.
- Taking r as the Earth’s radius for a point at latitude λ. The circle is about the axis, so $r = R\cos\lambda$.