A-LEVEL PHYSICS • EXERCISE SHEET

# 17.1.1 Proving a system is simple harmonic, and reading phase from a graph

Name: ______ Class: ______ Date: ______ **Total: 33 marks**

**KEY WORDS** displacement 位移 • simple harmonic motion 简谐运动 • acceleration 加速度 • frequency 频率 • amplitude 振幅

## Objective

Sheet 17.1 drilled the SHM equations on a system that was already known to be simple harmonic. This sheet is the part of 17.1 the exam spends its long questions on: **showing** that a real system oscillates with simple harmonic motion, naming the **assumptions** that make the proof work, and reading a **phase difference** 相位差 off a graph and quoting it in radians. The mass on a spring, the simple **pendulum** 单摆 and an unfamiliar system such as a floating cylinder are all the same four steps. This sheet covers syllabus 17.1.

**You must be able to:**

- use **displacement**, **amplitude** 振幅, **period**, **frequency**, **angular frequency** 角频率 and **phase difference**, and relate the period to both $f$ and $\omega$
- understand that **simple harmonic motion** occurs when the acceleration is **proportional to the displacement** from a fixed point and in the **opposite direction**
- use $a = -\omega^2 x$, and $x = x_0 \sin \omega t$ as a solution
- use $v = v_0 \cos \omega t$ and $v = \pm\,\omega\sqrt{x_0^2 - x^2}$
- analyse and interpret graphs of displacement, velocity and acceleration against time

## 1 Worked examples

Study these first. Each one shows a **method** that a question later on this sheet needs.

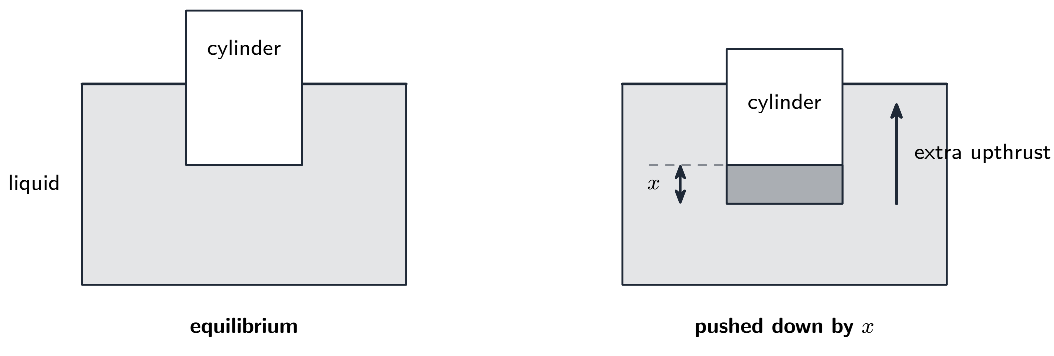
> ■ **"Show that the motion is simple harmonic" – always these four steps**
>
> The condition is a sentence and an equation, and an answer needs both:
>
> The **acceleration is proportional to the displacement** from a fixed point and is **always directed towards** that point, so $a = -\omega^2 x$.

| step | what to write |
|---|---|
| **1** | find the **resultant force** when the object is displaced by $x$, in terms of $x$ |
| **2** | apply $F = ma$ to get $a$ in terms of $x$ |
| **3** | the result has the form $a = -(\text{a constant})\,x$, so compare with $a = -\omega^2 x$ |
| **4** | read off $\omega^2$, then $T = \dfrac{2\pi}{\omega}$ |

| | mass on a spring | simple pendulum |
|---|---|---|
| 1 | $F = -kx$ | $F = -mg\sin\theta \approx -mg\dfrac{x}{L}$ for small $\theta$ |
| 2 | $a = -\dfrac{k}{m}x$ | $a = -\dfrac{g}{L}x$ |
| 3 | $\omega^2 = \dfrac{k}{m}$ | $\omega^2 = \dfrac{g}{L}$ |
| 4 | $T = 2\pi\sqrt{\dfrac{m}{k}}$ | $T = 2\pi\sqrt{\dfrac{L}{g}}$ |

- The **minus sign is the physics**, not a bookkeeping detail: it is the statement that the force is a **restoring** force.
- The **mass cancels** for the pendulum, which is why its period does not depend on the mass of the bob. For the spring it does not cancel, which is why the mass matters there.
- The pendulum step 1 uses $\sin\theta \approx \theta$, so the motion is only **approximately** simple harmonic, and only for **small angles**.

## The same four steps on a system you have never seen




The extra volume submerged is $Ax$, so the extra upthrust is $\rho Axg$, acting *upwards* while the displacement is *downwards*. That is a restoring force **proportional to the displacement and opposite to it** – the condition for simple harmonic motion – and Newton's second law then gives $a = -\left(\dfrac{\rho A g}{m}\right)x$, so $\omega^2 = \dfrac{\rho A g}{m}$.

*Worked example. A uniform cylinder of cross-sectional area $A$ and mass $m$ floats upright in a liquid of density $\rho$. It is pushed down a distance $x$ and released. Show that the motion is simple harmonic.*

1. At equilibrium the upthrust equals the weight, so the **resultant** force is only the **extra** upthrust from the extra volume $Ax$ that is now submerged.

$$F = -\rho A x g$$

   The minus sign says the force is upwards while the displacement is downwards.
2. Apply Newton's second law.

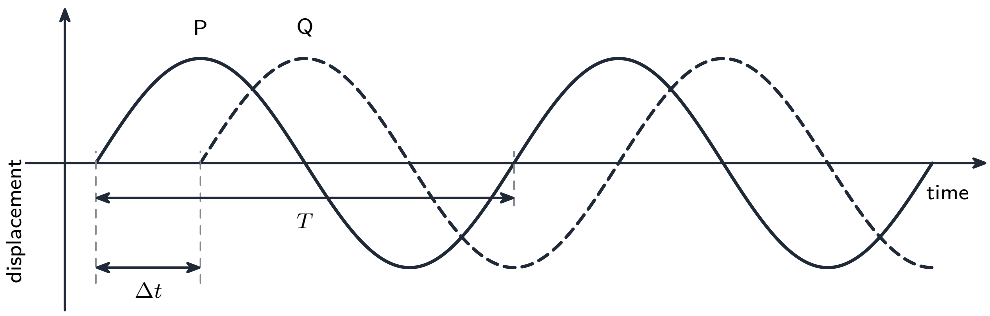
$$a = \frac{F}{m} = -\left(\frac{\rho A g}{m}\right) x$$

3. $\rho$, $A$, $g$ and $m$ are all constants, so $a$ is proportional to $x$ and opposite to it: the motion **is** simple harmonic.
4. Comparing with $a = -\omega^2 x$,

$$\omega^2 = \frac{\rho A g}{m}, \qquad T = 2\pi\sqrt{\frac{m}{\rho A g}}$$

- **The assumptions are worth marks.** Here: the cylinder stays **upright**, its sides are **vertical** so the extra volume really is $Ax$, the liquid's density is uniform, and **damping** (viscous drag) is negligible.

### ■ Phase difference, in radians




Read the **time lag** $\Delta t$ between the same feature on the two curves, then the **fraction of a cycle** $\Delta t/T$, then convert: $\varphi = 2\pi\,\dfrac{\Delta t}{T}$. Here $\Delta t = T/4$, so $\varphi = \pi/2$ rad, and Q *lags* P. Quote the answer in **radians** – "a quarter of a cycle" is only half the mark.

$$\varphi = 2\pi\,\frac{\Delta t}{T}$$

Read the **time lag** $\Delta t$ between the same feature on the two curves – two peaks, or two upward zero crossings – then the **fraction of a cycle**, then convert. The answer the mark scheme wants is in **radians**.

The three standard relationships inside one oscillation, which follow from differentiating $x = x_0 \sin \omega t$:

| pair | phase difference |
| --- | --- |
| **velocity** ahead of **displacement** | $\frac{\pi}{2}$ rad (a quarter of a cycle) |
| **acceleration** ahead of **velocity** | $\frac{\pi}{2}$ rad |
| **acceleration** and **displacement** | $\pi$ rad – exactly out of phase, which is the minus sign in $a = -\omega^2 x$ |

### Getting $\omega$, and then any speed you are asked for

- From a **graph**: measure the period $T$ and use $\omega = \frac{2\pi}{T}$.
- From an **acceleration-displacement** graph: it is a straight line through the origin of gradient $-\omega^2$, so $\omega = \sqrt{-\text{gradient}}$.
- Then $v_0 = \omega x_0$, $a_0 = \omega^2 x_0$, and at any displacement

$$v = \pm\,\omega\sqrt{x_0^2 - x^2}$$

- Work in **metres** throughout. A displacement quoted in millimetres is the single commonest arithmetic slip on this unit.

## 2 Practice

Now use the methods above.

**2.1** State the condition for simple harmonic motion in words, and write it as an equation. [2]

**2.2** A mass $m$ hangs on a spring of spring constant $k$ and is displaced a distance $x$ from equilibrium. Show that $a = -\frac{k}{m}x$, and hence write down the period. [3]

**2.3** Explain why the oscillation of a simple pendulum is only **approximately** simple harmonic. [2]

**2.4** A particle performs simple harmonic motion with amplitude 24 mm and period 0.80 s. Calculate its maximum acceleration. [3]

**2.5** Two oscillators have the same period of 0.60 s. Corresponding peaks on their displacement-time graphs are 0.15 s apart. Calculate the phase difference, in radians. [2]

**2.6** State the phase difference between displacement and velocity, and between displacement and acceleration, for a body in simple harmonic motion. [2]

## 3 Exam-style questions

**3.1** For a body in simple harmonic motion, a graph of acceleration $a$ against displacement $x$ is a straight line through the origin. The angular frequency $\omega$ is found from the gradient as [1]

| A | B |
|---|---|
| C | D |

**A** $\omega = \text{gradient}$
**B** $\omega = -\text{gradient}$
**C** $\omega = \sqrt{-\text{gradient}}$
**D** $\omega = \sqrt{\text{gradient}}$

**3.2** A uniform cylinder of cross-sectional area $A$ and mass $m$ floats upright in a liquid of density $\rho$. It is pushed vertically downwards a distance $x$ and released.

(a) Show that the resultant force on the cylinder is $-\rho A g x$. [2]

(b) Hence show that the motion is simple harmonic, and give an expression for $\omega^2$. [2]

(c) A cylinder of mass 0.45 kg and cross-sectional area $2.8 \times 10^{-3}$ m$^2$ floats in water of density 1000 kg m$^{-3}$. Calculate its period of oscillation. [2]

**3.3** A student measured the period $T$ of a simple pendulum for several lengths $L$ and plotted $T^2$ against $L$. The graph was a straight line through the origin with gradient 4.06 s$^2$ m$^{-1}$.

(a) Show that $T^2 = \dfrac{4\pi^2}{g} L$ for a simple pendulum. [2]

(b) Use the gradient to calculate a value for $g$. [2]

(c) State **two** assumptions made in deriving the expression in **(a)**. [2]

(d) The pendulum is set to a length of 0.65 m and released from a horizontal displacement of 35 mm. Calculate the maximum speed of the bob. [3]

(e) Calculate the speed of the bob when its displacement is 20 mm. [2]

(f) State and explain the effect on the period of doubling the mass of the bob. [1]

## Solutions

Each answer shows the steps, then the **result**.

### 2.1

- the **acceleration is proportional to the displacement** from a fixed point, and is always directed **towards** that point
- $a = -\omega^2 x$

### 2.2

- the restoring force from the spring is $F = -kx$
- apply Newton's second law

$$a = \frac{F}{m} = -\frac{k}{m}x$$

- comparing with $a = -\omega^2 x$ gives $\omega^2 = \dfrac{k}{m}$, so

$$T = \frac{2\pi}{\omega} = \mathbf{2}\pi\sqrt{\frac{\mathbf{m}}{\mathbf{k}}}$$

### 2.3

- the restoring force is $-mg\sin\theta$, which is proportional to $\sin\theta$ and **not** to the displacement itself
- only for **small angles** does $\sin\theta \approx \theta$, making the force proportional to $x$; for a large amplitude the approximation fails and the motion is no longer simple harmonic

### 2.4

- find $\omega$ from the period

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80\ \text{s}} = 7.85\ \text{rad s}^{-1}$$

- then $a_0 = \omega^2 x_0$, with the amplitude in **metres**

$$a_0 = (7.85\ \text{rad s}^{-1})^2 \times 0.024\ \text{m} = \mathbf{1.5}\ \text{m s}^{\mathbf{-2}}$$

### 2.5

- the lag is a fraction $\dfrac{\Delta t}{T} = \dfrac{0.15\ \text{s}}{0.60\ \text{s}} = \dfrac{1}{4}$ of a cycle

$$\varphi = 2\pi \times \frac{1}{4} = \frac{\pi}{\mathbf{2}}\ \text{rad (1.6 rad)}$$

### 2.6

- displacement and velocity: $\dfrac{\pi}{2}$ rad, with the velocity **ahead**
- displacement and acceleration: $\pi$ rad – exactly out of phase

### 3.1 C

- $a = -\omega^2 x$, so the gradient of the line is $-\omega^2$, which is negative
- therefore $\omega = \sqrt{-\text{gradient}}$. **A** and **B** forget the square root; **D** gives the square root of a negative number

**3.2**

(a)

- at equilibrium the upthrust already balances the weight, so the resultant force comes only from the **extra** upthrust
- the extra volume submerged is $Ax$, so the extra upthrust is $\rho Axg$, acting **upwards** while $x$ is measured **downwards**

$$F = -\rho Agx$$

(b)

- apply Newton's second law

$$a = \frac{F}{m} = -\left(\frac{\rho Ag}{m}\right)x$$

- $\rho$, $A$, $g$ and $m$ are constant, so $a \propto -x$: the motion is simple harmonic, with

$$\omega^2 = \frac{\rho Ag}{m}$$

(c)

- substitute, keeping units

$$\omega^2 = \frac{1000\ \text{kg m}^{-3} \times 2.8 \times 10^{-3}\ \text{m}^2 \times 9.81\ \text{m s}^{-2}}{0.45\ \text{kg}} = 61.0\ \text{s}^{-2}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{7.81\ \text{rad s}^{-1}} = \mathbf{0.80}\ \text{s}$$

**3.3**

(a)

- for a small angle the restoring force is $-mg\dfrac{x}{L}$, so $a = -\dfrac{g}{L}x$ and $\omega^2 = \dfrac{g}{L}$
- therefore

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \quad\Longrightarrow\quad T^2 = \frac{4\pi^2}{g}L$$

(b)

- the gradient is $\dfrac{4\pi^2}{g}$, so rearrange for $g$

$$g = \frac{4\pi^2}{\text{gradient}} = \frac{4\pi^2}{4.06\ \text{s}^2\ \text{m}^{-1}} = \mathbf{9.7}\ \text{m s}^{-2}$$

(c)

- any **two** of: the angle of swing is **small**, so that $\sin\theta \approx \theta$; the mass of the **string is negligible**; the bob behaves as a **point mass**; **air resistance** and other damping are negligible; the string does not stretch

(d)

- find the period from the graph, then $\omega$

$$T^2 = 4.06 \text{ s}^2 \text{ m}^{-1} \times 0.65 \text{ m} = 2.64 \text{ s}^2, \qquad T = 1.62 \text{ s}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{1.62 \text{ s}} = 3.87 \text{ rad s}^{-1}$$

- then $v_0 = \omega x_0$, with the amplitude in metres

$$v_0 = 3.87 \text{ rad s}^{-1} \times 0.035 \text{ m} = \mathbf{0.14} \text{ m s}^{-1}$$

(e)

- use $v = \omega\sqrt{x_0^2 - x^2}$

$$v = 3.87 \text{ rad s}^{-1} \times \sqrt{(0.035 \text{ m})^2 - (0.020 \text{ m})^2} = 3.87 \times 0.0287 \text{ m} = \mathbf{0.11} \text{ m s}^{-1}$$

(f)

- **no change**: the mass cancelled in $a = -\frac{g}{L}x$, so the period depends only on $L$ and $g$