Exercise walk-through

17.1.1

Proving a system is simple harmonic, and reading phase from a graph

A-Level Physics

You must be able to

- use **displacement**, **amplitude 振幅**, **period**, **frequency**, **angular frequency 角频率** and **phase difference**, and relate the period to both f and ω
- understand that **simple harmonic motion** occurs when the acceleration is **proportional to the displacement** from a fixed point and in the **opposite direction**
- use $a = -\omega^2 x$, and $x = x_0 \sin \omega t$ as a solution
- use $v = v_0 \cos \omega t$ and $v = \pm \omega \sqrt{x_0^2 - x^2}$
- analyse and interpret graphs of displacement, velocity and acceleration against time

1 The method

Method — "Show that the motion is simple harmonic" – always these four steps

The condition is a sentence and an equation, and an answer needs both:

> *The* ***acceleration is proportional to the displacement*** *from a fixed point and is* ***always directed towards*** *that point, so* $a = -\omega^2 x$.

step	what to write
1	find the **resultant force** when the object is displaced by x, in terms of x
2	apply $F = ma$ to get a in terms of x
3	the result has the form $a = -(\text{a constant})\,x$, so compare with $a = -\omega^2 x$
4	read off ω^2, then $T = \dfrac{2\pi}{\omega}$

Method — "Show that the motion is simple harmonic" – always these four steps

	mass on a spring	**simple pendulum**
1	$F = -kx$	$F = -mg\sin\theta \approx -mg\frac{x}{L}$ for small θ
2	$a = -\frac{k}{m}x$	$a = -\frac{g}{L}x$
3	$\omega^2 = \frac{k}{m}$	$\omega^2 = \frac{g}{L}$
4	$T = 2\pi\sqrt{\frac{m}{k}}$	$T = 2\pi\sqrt{\frac{L}{g}}$

- The **minus sign is the physics**, not a bookkeeping detail: it is the statement that the force is a **restoring** force.
- The **mass cancels** for the pendulum, which is why its period does not depend on the mass of the bob. For the spring it does not cancel, which is why the mass matters there.
- The pendulum step 1 uses $\sin\theta \approx \theta$, so the motion is only **approximately** simple harmonic, and only for **small angles**.

Method — The same four steps on a system you have never seen

Worked example. A uniform cylinder of cross-sectional area A and mass m floats upright in a liquid of density ρ. It is pushed down a distance x and released. Show that the motion is simple harmonic.

1. At equilibrium the upthrust equals the weight, so the **resultant** force is only the **extra** upthrust from the extra volume Ax that is now submerged.

$$F = -\rho A x g$$

   The minus sign says the force is upwards while the displacement is downwards.

2. Apply Newton's second law.

$$a = \frac{F}{m} = -\left(\frac{\rho A g}{m}\right) x$$

Method — The same four steps on a system you have never seen

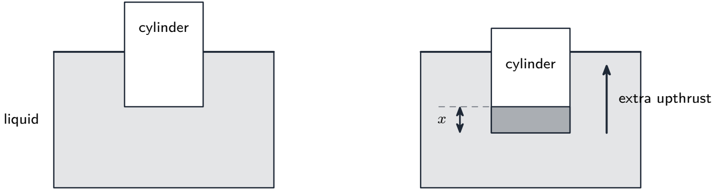
3. ρ, A, g and m are all constants, so a is proportional to x and opposite to it: the motion **is** simple harmonic.
4. Comparing with $a = -\omega^2 x$,

$$\omega^2 = \frac{\rho A g}{m}, \qquad T = 2\pi\sqrt{\frac{m}{\rho A g}}$$

- **The assumptions are worth marks.** Here: the cylinder stays **upright**, its sides are **vertical** so the extra volume really is Ax, the liquid's density is uniform, and **damping** (viscous drag) is negligible.

Method — The same four steps on a system you have never seen

Show that the motion is simple harmonic



equilibrium **pushed down by x**

The extra volume submerged is Ax, so the extra upthrust is ρAxg, acting *upwards* while the displacement is *downwards*. That is a restoring force **proportional to the displacement and opposite to it** – the condition for simple harmonic motion – and Newton's second law then gives $a = -\left(\frac{\rho Ag}{m}\right)x$, so $\omega^2 = \frac{\rho Ag}{m}$.

Method — Phase difference, in radians

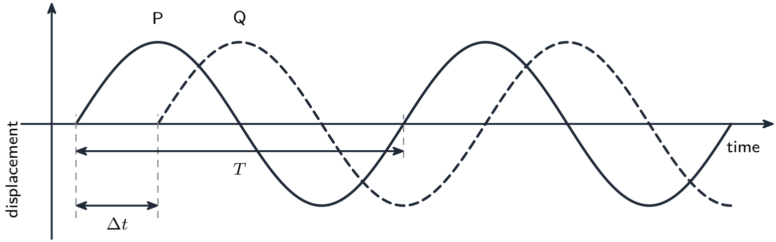
$$\varphi = 2\pi \, \frac{\Delta t}{T}$$

Read the **time lag** Δt between the same feature on the two curves – two peaks, or two upward zero crossings – then the **fraction of a cycle**, then convert. The answer the mark scheme wants is in **radians**.
The three standard relationships inside one oscillation, which follow from differentiating $x = x_0 \sin \omega t$:

pair	**phase difference**
velocity ahead of **displacement**	$\frac{\pi}{2}$ rad (a quarter of a cycle)
acceleration ahead of **velocity**	$\frac{\pi}{2}$ rad
acceleration and **displacement**	π rad – exactly out of phase, which is the minus sign in $a = -\omega^2 x$

Method — Phase difference, in radians

[= 2 , t T] Read the time lag ... between the same feature on the two curves – two peaks, or two upward zero...



Read the **time lag** Δt between the same feature on the two curves, then the **fraction of a cycle** $\Delta t/T$, then convert: $\varphi = 2\pi \dfrac{\Delta t}{T}$. Here $\Delta t = T/4$, so $\varphi = \pi/2$ rad, and Q *lags* P. Quote the answer in **radians** – "a quarter of a cycle" is only half the mark.

Method — Getting ω, and then any speed you are asked for

- From a **graph**: measure the period T and use $\omega = \dfrac{2\pi}{T}$.
- From an **acceleration-displacement** graph: it is a straight line through the origin of gradient $-\omega^2$, so $\omega = \sqrt{-\text{gradient}}$.
- Then $v_0 = \omega x_0$, $a_0 = \omega^2 x_0$, and at any displacement

$$v = \pm\omega\sqrt{x_0^2 - x^2}$$

- Work in **metres** throughout. A displacement quoted in millimetres is the single commonest arithmetic slip on this unit.

2 Practice

Practice 2.1 ▪ 2 marks

State the condition for simple harmonic motion in words, and write it as an equation.

Solution 2.1

Method: "Show that the motion is simple harmonic" – always these four steps — p.4

Worked steps

- the **acceleration is proportional to the displacement** from a fixed point, and is always directed **towards** that point **B1**
- $a = -\omega^2 x$ **B1**

Practice 2.2 ▪ 3 marks

A mass m hangs on a spring of spring constant k and is displaced a distance x from equilibrium. Show that $a = -\frac{k}{m}x$, and hence write down the period.

Solution 2.2

Method: "Show that the motion is simple harmonic" – always these four steps — p.4

Worked steps

- the restoring force from the spring is $F = -kx$ **B1**
- apply Newton's second law **M1**

$$a = \frac{F}{m} = -\frac{k}{m}x$$

- comparing with $a = -\omega^2 x$ gives $\omega^2 = \dfrac{k}{m}$, so

$$T = \frac{2\pi}{\omega} = \mathbf{2}\pi\sqrt{\frac{\mathbf{m}}{\mathbf{k}}}$$ **A1**

Practice 2.3 ▪ 2 marks

Explain why the oscillation of a simple pendulum is only **approximately** simple harmonic.

Solution 2.3

Method: "Show that the motion is simple harmonic" – always these four steps — p.4

Worked steps

- the restoring force is $-mg\sin\theta$, which is proportional to $\sin\theta$ and **not** to the displacement itself **B1**
- only for **small angles** does $\sin\theta \approx \theta$, making the force proportional to x; for a large amplitude the approximation fails and the motion is no longer simple harmonic **B1**

Practice 2.4 ▪ 3 marks

A particle performs simple harmonic motion with amplitude 24 mm and period 0.80 s. Calculate its maximum acceleration.

Solution 2.4

Method: "Show that the motion is simple harmonic" – always these four steps — p.4

Worked steps

- find ω from the period **M1**

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80\ \mathrm{s}} = 7.85\ \mathrm{rad\ s^{-1}}$$

- then $a_0 = \omega^2 x_0$, with the amplitude in **metres** **M1**

$$a_0 = \omega^2 x_0 = (7.85\ \mathrm{rad\ s^{-1}})^2 \times 0.024\ \mathrm{m} = \mathbf{1.5\ m\ s^{-2}}$$ **A1**

Practice 2.5 ▪ 2 marks

Two oscillators have the same period of 0.60 s. Corresponding peaks on their displacement-time graphs are 0.15 s apart. Calculate the phase difference, in radians.

Solution 2.5

Method: Phase difference, in radians — p.9

Worked steps

- the lag is a fraction $\frac{\Delta t}{T} = \frac{0.15 \text{ s}}{0.60 \text{ s}} = \frac{1}{4}$ of a cycle **M1**

$$\varphi = 2\pi \times \frac{1}{4} = \frac{\pi}{\mathbf{2}} \text{ rad } (1.6 \text{ rad})$$ **A1**

Practice 2.6 ▪ 2 marks

State the phase difference between displacement and velocity, and between displacement and acceleration, for a body in simple harmonic motion.

Solution 2.6

Method: Phase difference, in radians — p.9

Worked steps

- displacement and velocity: $\dfrac{\pi}{2}$ rad, with the velocity **ahead** **B1**
- displacement and acceleration: π rad – exactly out of phase **B1**

3 Exam-style

Exam-style 3.1 ▪ 1 mark

For a body in simple harmonic motion, a graph of acceleration a against displacement x is a straight line through the origin. The angular frequency ω is found from the gradient as

A $\omega = \text{gradient}$

B $\omega = -\text{gradient}$

C $\omega = \sqrt{-\text{gradient}}$

D $\omega = \sqrt{\text{gradient}}$

Solution 3.1

Method: Getting ω, and then any speed you are asked for — p.11

A $\omega = \text{gradient}$

B $\omega = -\text{gradient}$

C $\omega = \sqrt{-\text{gradient}}$ ✓

D $\omega = \sqrt{\text{gradient}}$

Worked steps

C

- $a = -\omega^2 x$, so the gradient of the line is $-\omega^2$, which is negative **B1**
- therefore $\omega = \sqrt{-\text{gradient}}$. **A** and **B** forget the square root; **D** gives the square root of a negative number

Exam-style 3.2 ▪ 6 marks

A uniform cylinder of cross-sectional area A and mass m floats upright in a liquid of density ρ. It is pushed vertically downwards a distance x and released.

Exam-style 3.2 (a) ▪ 2 marks

Show that the resultant force on the cylinder is $-\rho A g x$.

Solution 3.2 (a)

Method: The same four steps on a system you have never seen — p.6

Worked steps

- at equilibrium the upthrust already balances the weight, so the resultant force comes only from the **extra** upthrust **B1**
- the extra volume submerged is Ax, so the extra upthrust is ρAxg, acting **upwards** while x is measured **downwards** **B1**

$$F = -\rho Agx$$

Exam-style 3.2 (b) ▪ 2 marks

Hence show that the motion is simple harmonic, and give an expression for ω^2.

Solution 3.2 (b)

Worked steps

- apply Newton's second law **M1**

$$a = \frac{F}{m} = -\left(\frac{\rho A g}{m}\right) x$$

- ρ, A, g and m are constant, so $a \propto -x$: the motion is simple harmonic, with

$$\omega^2 = \frac{\rho A g}{m}$$ **A1**

Exam-style 3.2 (c) ▪ 2 marks

A cylinder of mass 0.45 kg and cross-sectional area 2.8×10^{-3} m^2 floats in water of density 1000 kg m^{-3}. Calculate its period of oscillation.

Solution 3.2 (c)

Worked steps

- substitute, keeping units **M1**

$$\omega^2 = \frac{\rho A g}{m} = \frac{1000 \text{ kg m}^{-3} \times 2.8 \times 10^{-3} \text{ m}^2 \times 9.81 \text{ m s}^{-2}}{0.45 \text{ kg}} = 61.0 \text{ s}^{-2}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{7.81 \text{ rad s}^{-1}} = \mathbf{0.80} \text{ s}$$ **A1**

Exam-style 3.3 ▪ 12 marks

A student measured the period T of a simple pendulum for several lengths L and plotted T^2 against L. The graph was a straight line through the origin with gradient 4.06 s^2 m^{-1}.

Exam-style 3.3 (a) ▪ 2 marks

Show that $T^2 = \dfrac{4\pi^2}{g}L$ for a simple pendulum.

Solution 3.3 (a)

Method: Getting ω, and then any speed you are asked for — p.11

Worked steps

- for a small angle the restoring force is $-mg\frac{x}{L}$, so $a = -\frac{g}{L}x$ and $\omega^2 = \frac{g}{L}$ **B1**
- therefore **B1**

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \quad \Longrightarrow \quad T^2 = \frac{4\pi^2}{g}L$$

Exam-style 3.3 (b) ▪ 2 marks

Use the gradient to calculate a value for g.

Solution 3.3 (b)

Worked steps

- the gradient is $\dfrac{4\pi^2}{g}$, so rearrange for g **M1**

$$g = \frac{4\pi^2}{\text{gradient}} = \frac{4\pi^2}{4.06\ \text{s}^2\ \text{m}^{-1}} = \mathbf{9.7}\ \text{m s}^{\mathbf{-2}}$$ **A1**

Exam-style 3.3 (c) ▪ 2 marks

State **two** assumptions made in deriving the expression in **(a)**.

Solution 3.3 (c)

Worked steps

- any **two** of: the angle of swing is **small**, so that $\sin\theta \approx \theta$; the mass of the **string is negligible**; the bob behaves as a **point mass**; **air resistance** and other damping are negligible; the string does not stretch **B1** **B1**

Exam-style 3.3 (d) ▪ 3 marks

The pendulum is set to a length of 0.65 m and released from a horizontal displacement of 35 mm. Calculate the maximum speed of the bob.

Solution 3.3 (d)

Worked steps

- find the period from the graph, then ω **M1**

$$T^2 = \text{gradient} \times l = 4.06 \text{ s}^2 \text{ m}^{-1} \times 0.65 \text{ m} = 2.64 \text{ s}^2, \qquad T = 1.62 \text{ s}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{1.62 \text{ s}} = 3.87 \text{ rad s}^{-1}$$ **M1**

- then $v_0 = \omega x_0$, with the amplitude in metres

$$v_0 = \omega x_0 = 3.87 \text{ rad s}^{-1} \times 0.035 \text{ m} = \mathbf{0.14} \text{ m s}^{-1}$$ **A1**

Exam-style 3.3 (e) ▪ 2 marks

Calculate the speed of the bob when its displacement is 20 mm.

Solution 3.3 (e)

Worked steps

- use $v = \omega\sqrt{x_0^2 - x^2}$ **M1**

$$v = \omega\sqrt{x_0^2 - x^2} = 3.87 \text{ rad s}^{-1} \times \sqrt{(0.035 \text{ m})^2 - (0.020 \text{ m})^2}$$

$$v = 3.87 \text{ rad s}^{-1} \times 0.0287 \text{ m} = \mathbf{0.11} \textbf{ m s}^{\mathbf{-1}}$$ **A1**

Exam-style 3.3 (f) ▪ 1 mark

State and explain the effect on the period of doubling the mass of the bob.

Solution 3.3 (f)

Worked steps

- **no change**: the mass cancelled in $a = -\frac{g}{L}x$, so the period depends only on L and g **B1**