Exercise walk-through

15.3.1

Deriving pV = one third Nm mean-square speed, and what it says about temperature

A-Level Physics

You must be able to

- state the basic **assumptions** 假设 of the **kinetic theory of gases** 气体分子运动论
- explain how molecular movement causes the pressure of a gas, and **derive** and use $pV = \frac{1}{3}Nm\langle c^2 \rangle$
- understand that the root-mean-square speed is $c_{\text{r.m.s.}} = \sqrt{\langle c^2 \rangle}$
- compare $pV = \frac{1}{3}Nm\langle c^2 \rangle$ with $pV = NkT$ to deduce that the mean translational kinetic energy of a molecule is $\frac{3}{2}kT$, and use that expression

1 The method

Method — The assumptions, and what each one is for

assumption	**why the derivation needs it**
the gas contains a **large number** of molecules in **random** motion	so that averages are meaningful and no direction is special
the **volume of the molecules** is negligible compared with the volume of the container	so the molecules travel the full distance L between walls
collisions are **perfectly elastic**	so the speed is unchanged and the momentum change is exactly $2mv_x$
the time of a **collision** is negligible compared with the time between collisions	so the force can be averaged over the round trip
there are **no forces** between molecules except during collisions	so a molecule travels in a straight line at constant velocity between walls

Method — The assumptions, and what each one is for

- Quote them as **properties of the model**, not of a real gas. The exam wants the assumptions, not a description of a gas.

Method — The derivation, five steps

1. One collision reverses v_x, so the molecule's momentum changes from $+mv_x$ to $-mv_x$: a change of $2mv_x$. By Newton's third law the wall receives the same.
2. The molecule crosses the box and comes back before hitting the same wall again, a distance $2L$, so the time between collisions is $\Delta t = \dfrac{2L}{v_x}$.
3. Force is rate of change of momentum:

$$F = \frac{2mv_x}{2L/v_x} = \frac{mv_x^2}{L}$$

4. Pressure is force per unit area, and the wall has area L^2, so one molecule contributes $\dfrac{mv_x^2}{L^3}$. For N molecules, using the **mean** of v_x^2:

$$p = \frac{Nm\langle v_x^2\rangle}{V}$$

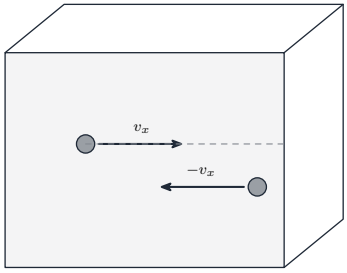
Method — The derivation, five steps

5. The motion is **random**, so no direction is special and $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$. Since $c^2 = v_x^2 + v_y^2 + v_z^2$, this gives $\langle v_x^2 \rangle = \frac{1}{3}\langle c^2 \rangle$, and so

$$pV = \tfrac{1}{3} Nm \langle c^2 \rangle$$

- $\langle c^2 \rangle$ is the **mean-square speed** – square every molecule's speed, then average. The **root-mean-square speed** is $c_{\text{r.m.s.}} = \sqrt{\langle c^2 \rangle}$, and it is *not* the mean speed.

Method — The derivation, five steps



side L, so volume $V = L^3$

1 one collision with this wall reverses v_x, so the molecule's momentum changes by $2mv_x$, and by Newton's third law the wall receives $2mv_x$

2 it next hits the same wall after travelling $2L$, so the time between collisions is $\Delta t = \dfrac{2L}{v_x}$

3 so the average force on the wall is $F = \dfrac{2mv_x}{2L/v_x} = \dfrac{mv_x^2}{L}$

4 Pressure is F/L^2, so one molecule contributes $\dfrac{mv_x^2}{L^3}$; adding N molecules gives $p = \dfrac{Nm\langle v_x^2\rangle}{V}$. **5** No direction is special, so $\langle v_x^2\rangle = \frac{1}{3}\langle c^2\rangle$, and therefore $pV = \frac{1}{3}Nm\langle c^2\rangle$.

Method — From the equation to the temperature

Compare the derived result with the ideal-gas equation written per molecule:

$$pV = \tfrac{1}{3}Nm\langle c^2\rangle \qquad \text{and} \qquad pV = NkT$$

so $\tfrac{1}{3}Nm\langle c^2\rangle = NkT$, and multiplying both sides by $\tfrac{3}{2N}$:

$$\tfrac{1}{2}m\langle c^2\rangle = \tfrac{3}{2}kT$$

- The left side is the **mean translational kinetic energy** of one molecule. So **thermodynamic temperature is a measure of the mean translational kinetic energy of the molecules** – that sentence is the point of the whole topic.
- It contains **no reference to the gas**: at the same temperature, every ideal gas has the same mean molecular kinetic energy, whatever the molecules are made of.

Method — Three consequences worth having ready

$$\langle c^2 \rangle = \frac{3kT}{m} \qquad \Longrightarrow \qquad c_{\text{r.m.s.}} = \sqrt{\frac{3kT}{m}}$$

question	**answer**
two gases at the **same temperature**	same mean kinetic energy, so $c_{\text{r.m.s.}} \propto \frac{1}{\sqrt{m}}$: the **lighter** molecule is faster
the same gas at **double** the temperature	$c_{\text{r.m.s.}} \propto \sqrt{T}$, so the speed rises by a factor of $\sqrt{2}$, not 2
heating at **constant volume**	the molecules move faster, so they hit the walls **more often** *and* with a **larger momentum change** each time; the average force, and so the pressure, rises

Method — Three consequences worth having ready

- Both halves of that last answer are needed. “They move faster so they hit harder” is worth one mark of two.

2 Practice

Practice 2.1 ▪ 3 marks

State **three** assumptions of the kinetic theory of gases.

Solution 2.1

Method: Three consequences worth having ready — p.10

Worked steps

- any **three** of: a large number of molecules in **random** motion; the **volume of the molecules** is negligible compared with the volume of the container; collisions are **perfectly elastic**; the duration of a collision is negligible compared with the time between collisions; there are **no forces** between molecules except during collisions B1 B1 B1

Practice 2.2 ▪ 2 marks

Explain why the momentum change of a molecule rebounding from a wall is $2mv_x$ and not mv_x.

Solution 2.2

Method: The derivation, five steps — p.6

Worked steps

- momentum is a **vector**: the molecule arrives with $+mv_x$ and leaves with $-mv_x$ B1
- the **change** is $(-mv_x) - (+mv_x) = -2mv_x$, so its magnitude is $2mv_x$ B1

Practice 2.3 ▪ 2 marks

Show that the time between successive collisions of one molecule with the **same** wall of a cubical box of side L is $\frac{2L}{v_x}$.

Solution 2.3

Method: The derivation, five steps — p.6

Worked steps

- after rebounding, the molecule must cross the box and return before it can hit the same wall again, a distance of $2L$ **M1**
- its component of velocity in that direction is v_x and is unchanged in magnitude, so

$$\Delta t = \frac{\text{distance}}{\text{speed}} = \frac{2L}{v_x}$$ **A1**

Practice 2.4 ▪ 2 marks

State what is meant by the **mean-square speed** and by the **root-mean-square speed**.

Solution 2.4

Method: The derivation, five steps — p.6

Worked steps

- the **mean-square speed** $\langle c^2 \rangle$ is the average of the squares of the speeds of all the molecules B1
- the **root-mean-square speed** is the square root of that, $c_{\text{r.m.s.}} = \sqrt{\langle c^2 \rangle}$ B1

Practice 2.5 ▪ 2 marks

Explain why $\langle v_x^2 \rangle = \frac{1}{3}\langle c^2 \rangle$.

Solution 2.5

Worked steps

- the motion is **random**, so no direction is preferred and $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$ B1
- since $c^2 = v_x^2 + v_y^2 + v_z^2$, taking means gives $\langle c^2 \rangle = 3\langle v_x^2 \rangle$, so $\langle v_x^2 \rangle = \frac{1}{3}\langle c^2 \rangle$ B1

3 Exam-style

Exam-style 3.1 ▪ 1 mark

The thermodynamic temperature of an ideal gas is doubled. The root-mean-square speed of its molecules is multiplied by

A $\sqrt{2}$

B 2

C 4

D $\frac{1}{\sqrt{2}}$

Solution 3.1

A $\sqrt{2}$ ✓

B 2

C 4

D $\dfrac{1}{\sqrt{2}}$

Worked steps

A

- $c_{\text{r.m.s.}} = \sqrt{\dfrac{3kT}{m}}$, so $c_{\text{r.m.s.}} \propto \sqrt{T}$ B1
- doubling T multiplies the speed by $\sqrt{2}$. **B** is the standard error – it is the mean *kinetic energy* that doubles, not the speed

Exam-style 3.2 ▪ 6 marks

The kinetic theory gives $pV = \frac{1}{3}Nm\langle c^2\rangle$, and the ideal-gas equation gives $pV = NkT$.

Exam-style 3.2 (a) ▪ 3 marks

Use these to show that the mean translational kinetic energy of a molecule is $\frac{3}{2}kT$.

Solution 3.2 (a)

Method: From the equation to the temperature — p.9

Worked steps

- both expressions equal pV, so set them equal **M1**

$$\tfrac{1}{3}Nm\langle c^2\rangle = NkT$$

- the N cancels **M1**

$$\tfrac{1}{3}m\langle c^2\rangle = kT$$

- multiply both sides by $\frac{3}{2}$

$$\tfrac{1}{2}m\langle c^2\rangle = \tfrac{3}{2}kT$$ **A1**

Exam-style 3.2 (b) ▪ 2 marks

Calculate the mean translational kinetic energy of a molecule at $27\ ^{\circ}\text{C}$. Take $k = 1.38 \times 10^{-23}$ J K^{-1}.

Solution 3.2 (b)

Worked steps

- convert the temperature to kelvin, $T = 300$ K **M1**

$$E = \tfrac{3}{2}kT = 1.5 \times 1.38 \times 10^{-23}\ \text{J K}^{-1} \times 300\ \text{K} = \mathbf{6.2 \times 10^{-21}}\ \text{J} \quad \textbf{A1}$$

Exam-style 3.2 (c) ▪ 1 mark

State what this result shows about the meaning of thermodynamic temperature.

Solution 3.2 (c)

Worked steps

- thermodynamic temperature is a measure of the **mean translational kinetic energy** of the molecules, and it is the same for every ideal gas at that temperature B1

Exam-style 3.3 ▪ 13 marks

Helium, of molar mass 4.0 g mol^{-1}, and oxygen, of molar mass 32 g mol^{-1}, are both held at 300 K. Take $N_A = 6.02 \times 10^{23}$ mol^{-1} and $k = 1.38 \times 10^{-23}$ J K^{-1}.

Exam-style 3.3 (a) ▪ 2 marks

Calculate the mass of one helium atom.

Solution 3.3 (a)

Method: Three consequences worth having ready — p.10

Worked steps

- one mole contains N_A atoms **M1**

$$m = \frac{4.0 \times 10^{-3}\ \text{kg mol}^{-1}}{6.02 \times 10^{23}\ \text{mol}^{-1}} = \mathbf{6.6 \times 10^{-27}}\ \text{kg} \qquad \textbf{A1}$$

Exam-style 3.3 (b) ▪ 3 marks

Calculate the root-mean-square speed of a helium atom at 300 K.

Solution 3.3 (b)

Worked steps

- rearrange $\frac{1}{2}m\langle c^2\rangle = \frac{3}{2}kT$ M1

$$\langle c^2\rangle = \frac{3kT}{m} = \frac{3 \times 1.38 \times 10^{-23}\ \text{J K}^{-1} \times 300\ \text{K}}{6.6 \times 10^{-27}\ \text{kg}} = 1.87 \times 10^{6}\ \text{m}^2\ \text{s}^{-2} \quad \text{M1}$$

$$c_{\text{r.m.s.}} = \sqrt{1.87 \times 10^{6}\ \text{m}^2\ \text{s}^{-2}} = \mathbf{1.4 \times 10^{3}}\ \text{m s}^{-1} \quad \text{A1}$$

Exam-style 3.3 (c) ▪ 3 marks

Without calculating it exactly, state and explain how the root-mean-square speed of an oxygen molecule at the same temperature compares with that of a helium atom.

Solution 3.3 (c)

Worked steps

- at the same temperature both gases have the **same mean translational kinetic energy**, $\frac{3}{2}kT$ B1
- so $\frac{1}{2}m\langle c^2\rangle$ is the same for both, and $c_{\text{r.m.s.}} \propto \frac{1}{\sqrt{m}}$ B1
- an oxygen molecule has **eight times** the mass, so its r.m.s. speed is $\frac{1}{\sqrt{8}}$ of the helium value – about **2.8 times smaller**, roughly 480 m s^{-1} B1

Exam-style 3.3 (d) ▪ 2 marks

The helium is now heated to 600 K. State and explain the effect on the root-mean-square speed of its atoms.

Solution 3.3 (d)

Worked steps

- the r.m.s. speed **increases by a factor of** $\sqrt{2}$, to about 1.9×10^3 m s^{-1} **B1**
- because $c_{\text{r.m.s.}} \propto \sqrt{T}$, and the thermodynamic temperature has doubled **B1**

Exam-style 3.3 (e) ▪ 3 marks

Use the kinetic theory to explain why the pressure of a fixed mass of gas increases when it is heated at constant volume.

Solution 3.3 (e)

Worked steps

- heating raises the mean translational kinetic energy, so the molecules move **faster** on average B1
- they therefore hit the walls **more frequently** B1
- and each collision produces a **greater change of momentum**; both effects raise the average force on the walls, and with the area unchanged the pressure rises B1