

Exercise walk-through

13.3

Gravitational field of a point mass

A-Level Physics

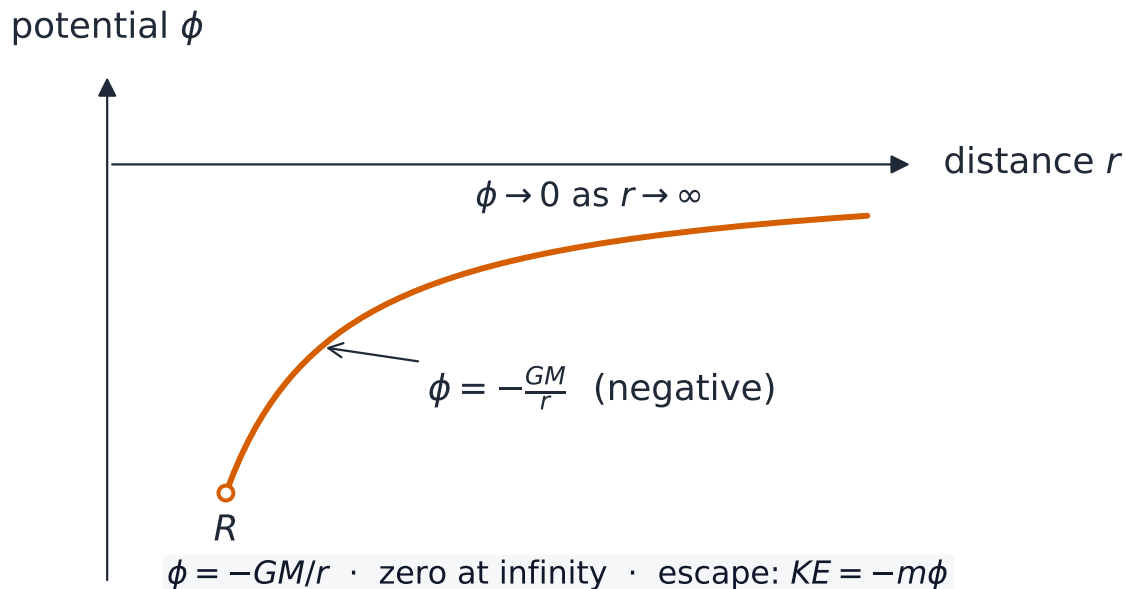
Gravitational potential

Potential infinity : is the work done **per unit mass** bringing a test mass in from

$$\phi = \frac{W}{m} = -\frac{GM}{r}, \quad [\phi] = \text{J kg}^{-1}$$

Zero at infinity, so ϕ is **negative** everywhere else – gravity did the work for you on the way in. ϕ is a **scalar**: just add them.

Gravitational potential, in detail



The gravitational potential well: $\phi = -GM/r$ is negative, rising to zero at infinity

Link with $\Delta E_p = mg\Delta h$

For small height changes near the surface, r barely changes, so $\Delta E_p \approx mg\Delta h$.

For large changes (a satellite moving to a higher orbit) use $-GMm/r$ at each radius and take the difference:

Potential energy and escape

A mass m sitting where the potential is ϕ :

$$E_p = m\phi = -\frac{GMm}{r}$$

Raising a satellite from r_1 to r_2 :

$$\Delta E_p = GMm \left(\frac{1}{r_1} - \frac{1}{r_2} \right) > 0$$

To escape, the kinetic energy must match the well's depth:

$$\frac{1}{2}mv_{\text{esc}}^2 = \frac{GMm}{r} \Rightarrow v_{\text{esc}} = \sqrt{\frac{2GM}{r}}$$

$mg\Delta h$ is only the **small-height** shortcut, where r hardly changes. Over orbital distances you must difference $-GMm/r$ at each radius.

Escape needs **no rocket left over**: reach v_{esc} and you coast to infinity, arriving with zero speed.

The energy of two point masses

Potential energy

The **gravitational potential energy** of two point masses is $E_p = -\frac{GMm}{r}$ – negative, because the zero is placed at infinity.

The constant

$G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$ is the **universal gravitational constant** – the same everywhere, which is what “universal” asserts.

Escape speed

Escape needs $\frac{1}{2}mv^2 \geq \frac{GMm}{r}$, so $v_{\text{esc}} = \sqrt{2GM/r}$ – independent of the escaping mass.

The minus sign is not decoration: it says work must be *done* to separate the masses, and it is what makes a bound orbit have negative total energy.

Exercises 2.1-2.3

2.1 Write the equation for the gravitational field strength g due to a point mass. [1]

2.2 State how g varies with distance r from a point mass. [1]

2.3 At the Earth's surface $g = 9.8 \text{ N kg}^{-1}$. Find g at a distance of $2R$ from the centre (a height R above the surface). [2]

Exercises 2.1·2.3

2.1 Write the equation for the gravitational field strength g due to a point mass. [1]

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- $g = GM/r^2$

2.2 State how g varies with distance r from a point mass. [1]

2.2

- $g \propto 1/r^2$ (inverse-square)

2.3 At the Earth's surface $g = 9.8 \text{ N kg}^{-1}$. Find g at a distance of $2R$ from the centre (a height R above the surface). [2]

2.3

- at $2R$, g is $1/2^2 = 1/4$ of the surface value

$$g = \frac{g_{\text{surface}}}{2^2} = \frac{9.8 \text{ N kg}^{-1}}{4} = \mathbf{2.45 \text{ N kg}^{-1}}$$

Exercises 2.4·3.1

2.4 Explain why g is **nearly constant** near the Earth's surface.

[1]

3.1 At **twice** the distance from a point mass, g becomes:

[1]

☐

A twice as large

B half as large

C one quarter as large

D four times as large

Exercises 2.4·3.1

2.4 Explain why g is **nearly constant** near the Earth's surface. [1]

2.4

- a small height change is **tiny compared with the Earth's radius**, so r (and hence $g \propto 1/r^2$) barely changes

3.1 At **twice** the distance from a point mass, g becomes: [1]

☐

A twice as large

B half as large

C one quarter as large

D four times as large

3.1 C

- $g \propto 1/r^2$, so at $2R$ the field is **one quarter**

Exercises 3.2·3.3

3.2 The Moon has mass 7.3×10^{22} kg and radius 1.7×10^6 m. Calculate g at its surface.
($G = 6.67 \times 10^{-11}$.) [3]

3.3 Derive $g = \frac{GM}{r^2}$ from Newton's law of gravitation and $g = F/m$. [2]

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- $g = GM/R^2$

$$g = \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2})(7.3 \times 10^{22} \text{ kg})}{(1.7 \times 10^6 \text{ m})^2} = \mathbf{1.7 \text{ N kg}^{-1}}$$

3.3 Derive $g = \frac{GM}{r^2}$ from Newton's law of gravitation and $g = F/m$. [2]

3.3

- force on a test mass m : $F = GMm/r^2$
- field strength $g = F/m = GM/r^2$

Exercises 3.4·3.5

3.4 A planet has the same mass as Earth but **twice** the radius. State its surface g as a fraction of Earth's. [1]

3.5 On the axes, **sketch** how the gravitational field strength g of a planet varies with distance r from its centre, for r from the surface ($r = R$) out to $r = 4R$. Mark the surface value g_0 . [2]

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- $g \propto 1/R^2$, so $1/4$ of Earth's

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3.5

- a curve **starting at** g_0 at $r = R$ and **falling** as an inverse square
- to $g_0/4$ at $2R$, $g_0/9$ at $3R$, $g_0/16$ at $4R$ – approaching (but never reaching) zero

Exercise 3.6 (a)

3.6 A planet of mass M and radius R has a gravitational field strength of g_0 at its surface. A satellite of mass m orbits it in a circle of radius r , where $r > R$.

(a) **State** why the gravitational field strength inside a **uniform** sphere behaves differently from the field outside it. [1]

$\sqrt{GM}$

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(a) **State** why the gravitational field strength inside a **uniform** sphere behaves differently from the field outside it. [1]

$\sqrt{GM/r}$

(a) outside the sphere all its mass may be treated as concentrated at the centre, giving $g \propto 1/x^2$; inside, only the mass **closer to the centre** than the point contributes, so the field falls towards zero at the centre instead of rising

Exercise 3.6 (b)

3.6 A planet of mass M and radius R has a gravitational field strength of g_0 at its surface. A satellite of mass m orbits it in a circle of radius r , where $r > R$.

(b) **Show that** the orbital speed of the satellite is $v = \sqrt{\frac{GM}{r}}$, and hence that it does **not** depend on the mass of the satellite. [3]

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(b) the gravitational force provides the centripetal force: $GMm/r^2 = mv^2/r$

- the satellite mass m appears on both sides and **cancels**
- $v^2 = GM/r$, so $v = \sqrt{GM/r}$

Exercise 3.6 (c)

3.6 A planet of mass M and radius R has a gravitational field strength of g_0 at its surface. A satellite of mass m orbits it in a circle of radius r , where $r > R$.

(c) **Derive** an expression for the period T of the orbit in terms of G , M and r , and state the name of the law it expresses. [3]

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(c) $T = 2\pi r/v = 2\pi r\sqrt{r/(GM)}$

- $T^2 = 4\pi^2 r^3/(GM)$
- that is $T^2 \propto r^3$, **Kepler's third law**

Exercise 3.6 (d)

3.6 A planet of mass M and radius R has a gravitational field strength of g_0 at its surface. A satellite of mass m orbits it in a circle of radius r , where $r > R$.

(d) A geostationary satellite orbits the Earth once every 24 hours. Taking $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$, **determine** the radius of its orbit. [3]

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(d) $T = 24 \times 3600 \text{ s} = 86\,400 \text{ s}$

$$r^3 = \frac{GMT^2}{4\pi^2} = \frac{(4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2})(86\,400 \text{ s})^2}{4\pi^2} = 7.56 \times 10^{22} \text{ m}^3$$

- $r = 4.2 \times 10^7 \text{ m}$

Exercise 3.6 (e)

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(e) **State** two further conditions a geostationary orbit must satisfy, beyond having the right period.

[2]

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(e) **State** two further conditions a geostationary orbit must satisfy, beyond having the right period. [2]

(e) it must orbit **above the equator**, in the plane of the equator

- and travel **west to east**, the same direction as the Earth's rotation