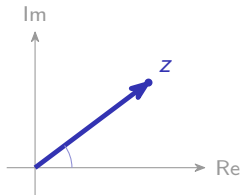


Pure Mathematics 3

A-Level Mathematics ■ Topic 3

A-Level Mathematics



What we will cover

- 1 Partial fractions & integration
- 2 Vectors
- 3 Differential equations
- 4 Complex numbers
- 5 Recap



1

Algebra and integration

Partial fractions and integration

Split a proper rational function into simpler pieces, then integrate term by term:

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Repeated factors need a squared denominator; a quadratic factor that will not factor stays as a linear numerator over it.

$\int \frac{1}{x} dx = \ln|x| + C$. Partial fractions turn many rational integrals into logs.

Partial fractions: check the degree first

the precondition	the top must be of lower degree than the bottom
if it is not	“top-heavy”: divide the polynomial first, giving a quotient 商 plus a proper remainder
distinct linear factors	$\frac{A}{ax+b} + \frac{B}{cx+d}$
a repeated factor	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
an irreducible quadratic	$\frac{Ax+B}{x^2+c}$ – a linear numerator over a quadratic
why bother	each piece integrates to a logarithm or a power, which the original fraction does not

Skipping the degree check is the commonest error, and it produces an answer that cannot be right because the shapes do not match.

The binomial expansion for a rational power

The **binomial expansion** 二项展开式 also works when the power is a fraction or is negative.

Logarithms, trigonometry and numerical methods (from Pure 2)

Subtopics 3.2, 3.3 and 3.6 are the same skills you met in Pure 2: the laws of logarithms with e^x and $\ln x$.

The three new trig functions are the **reciprocals** 倒数 of the familiar ones: the **secant** 正割 $\sec \theta = \frac{1}{\cos \theta}$.

Differentiation

The methods are those of Pure 2 (the product, quotient and chain rules, with parametric and implicit curves).

One derivative is added – the **inverse tangent** 反正切:

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}.$$

2

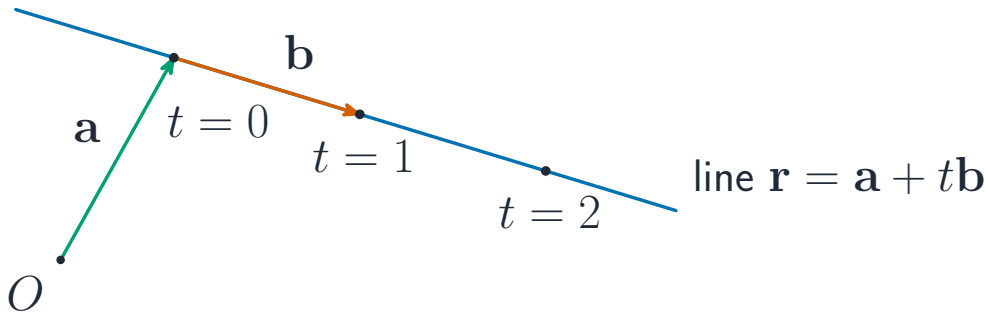
Vectors

Vectors

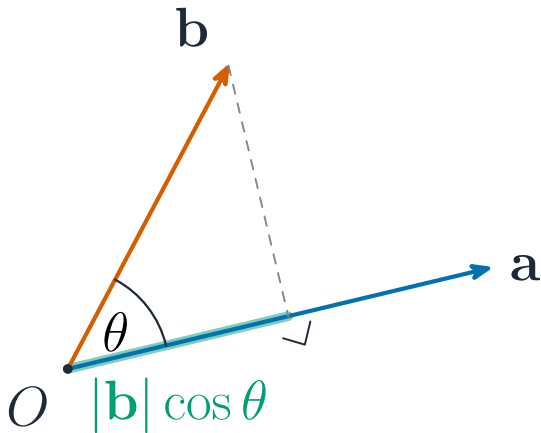
$|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$; a **unit vector** 单位向量 has length 1.

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \text{ (a line).}$$

Lines can be parallel, intersect, or be **skew** 异面直线.



The scalar product



$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos \theta \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3\end{aligned}$$

$$\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \text{perpendicular.}$$

e.g. angle between $(1, 2, 2)$ and $(2, 2, 1)$: $\cos \theta = \frac{8}{9}$, $\theta = 27.3^\circ$.

3

Differential equations

Differential equations

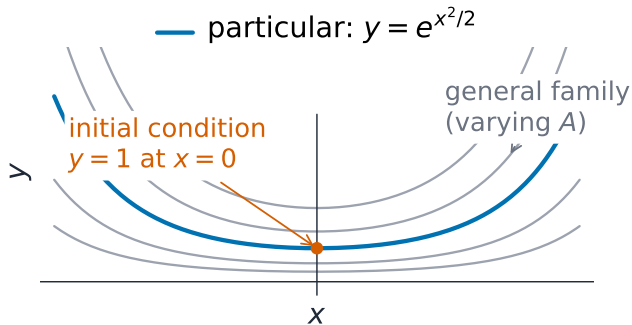
Separable 可分离变量: split y and x , then integrate both sides.

$$\frac{dy}{dx} = xy, y = 1 \text{ at } x = 0$$

$$\ln y = \frac{1}{2}x^2 + c \Rightarrow y = e^{x^2/2}$$

A condition fixes C (the **particular solution** 特解).

Differential equations, in detail



$\frac{dy}{dx} = xy$ gives $\ln y = \frac{1}{2}x^2 + c$; the condition $y = 1$ at $x = 0$ fixes $c = 0$, so
 $y = e^{x^2/2}$ – the **particular solution** 特解.

Separating the variables

a differential
equation 微分方程

links a quantity to its rate of change

separable 可分离

when it can be written $\frac{dy}{dx} = f(x)g(y)$

the method

$\int \frac{dy}{g(y)} = \int f(x) dx$ – all the y's one side, all the x's the other

the general
solution 通解

includes an arbitrary constant $+C$

the initial
condition 初始条件

one given pair of values fixes C , giving the particular solution

the modelling step

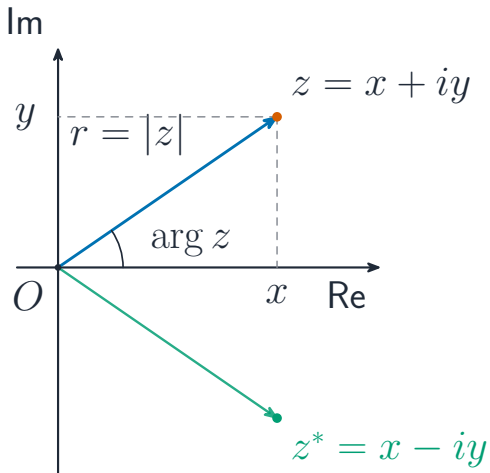
read the sentence into an equation: “proportional to” means $= k(\dots)$

Do not forget $+C$ before you apply the condition. Applying it afterwards to an equation with no constant leaves nothing to determine.

4

Complex numbers

Complex numbers



$z = x + iy$, $i^2 = -1$. **Conjugate** 共轭
 $z^* = x - iy$.

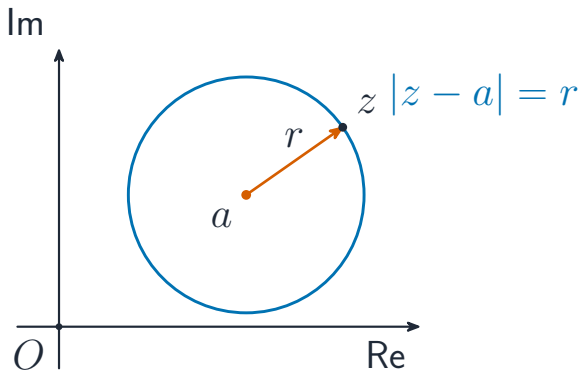
$$|z| = \sqrt{x^2 + y^2}, \quad z = re^{i\theta}$$

Divide by multiplying by the conjugate:

Simplify $(3 + i)/(1 - i)$

$$\frac{(3 + i)(1 + i)}{(1 - i)(1 + i)} = \frac{2 + 4i}{2} = 1 + 2i$$

Loci on the Argand diagram



A condition on z draws a **locus** 轨迹:

- $|z - a| = r$ – a circle
- $|z - a| = |z - b|$ – a perpendicular bisector
- $\arg(z - a) = \theta$ – a half-line

Complex numbers: the five things to know

Cartesian form 直角坐标形式

$z = x + iy$; x is the real part, y the **imaginary part** 虚部

the conjugate 共轭

$z^* = x - iy$. For a polynomial with **real** coefficients, non-real roots come in **conjugate pairs**

the modulus 模

$|z| = \sqrt{x^2 + y^2}$ – its distance from the origin

the argument 辐角

the angle the point makes, measured from the positive real axis

polar form 极坐标形式

$z = r(\cos \theta + i \sin \theta) = re^{i\theta}$ – multiplying multiplies the moduli and **adds** the arguments

on the Argand diagram

$|z|$ is the distance from O , $\arg z$ the angle, and z^* the **reflection in the real axis**

Choose the form by the operation: **Cartesian** for adding, **polar** for multiplying, dividing and taking powers.

Worked example – integration by parts

Find $\int x \cos x \, dx$.

By parts with $u = x$, $\frac{dv}{dx} = \cos x$ (so $v = \sin x$):

$$\begin{aligned}\int x \cos x \, dx &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C\end{aligned}$$

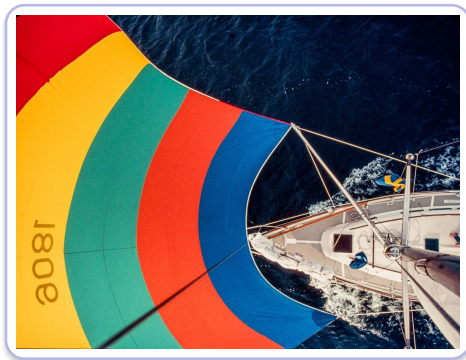
Choose u to be the part that gets **simpler** when differentiated – here $x \rightarrow 1$. The other part must be integrable.

Partial fractions

$$\frac{x+4}{(x+1)(x-2)} \xrightarrow{\text{split}} \frac{2}{x-2} - \frac{1}{x+1}$$

One fraction with a factorised bottom splits into a sum of simpler fractions.

Wind and water as vectors



Forces like wind and water are vectors – they have both size and direction.

The vectors vocabulary

**position
vector** 位置向量

gives a point's place from the origin

**displacement
vector** 位移向量

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ – from one point to another

the line

$\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$: a point on it, plus a direction

the scalar product

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$ – so it gives the angle

perpendicular

$\mathbf{a} \cdot \mathbf{b} = 0$

skew lines

in three dimensions two lines can be neither parallel nor intersecting – check both before concluding

“Do they meet?” is answered by solving for λ and μ in two components and testing the third. If it fails, the lines are skew.

Four more pieces of vocabulary

Integration by substitution

Integration by substitution 换元积分 reverses the chain rule – change the variable and the limits together.

Sign change and iteration

A **sign change** 符号变化 of a continuous f between a and b locates a root; an **iterative formula** 迭代公式 $x_{n+1} = g(x_n)$ then **converges** 收敛 to it when $|g'| < 1$ nearby.

Cosecant and cotangent

$\operatorname{cosec} \theta = 1/\sin \theta$ and $\cot \theta = 1/\tan \theta$ – each undefined wherever its denominator vanishes.

Magnitude

The **magnitude** 模 of a vector is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$; dividing by it gives the unit vector.

State the interval, the two signs *and* the continuity. A sign change without continuity is an incomplete argument.

Exam tips

- Split a rational function into **partial fractions** before integrating or expanding.
- Choose the right **integration technique** (substitution, by parts, or partial fractions) from the form of the integrand.
- Use the scalar (dot) product for the **angle between vectors** and to test for perpendicularity; write a line as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.
- Give **complex numbers** in the form asked for (Cartesian or modulus-argument) and show them on an Argand diagram.

5

Recap

Topic 3 – key takeaways

1

Integration

partial fractions, by parts,
substitution

2

Vectors

$\mathbf{r} = \mathbf{a} + t\mathbf{b}$; $\mathbf{a} \cdot \mathbf{b} = 0$ perpendicular

3

Diff. equations

separate variables; a condition
fixes C

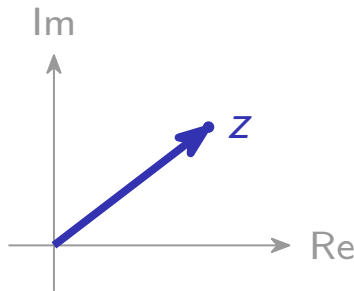
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Complex

$z = x + iy = re^{i\theta}$; loci on the
Argand plane

Check yourself

- 1 What does $\mathbf{a} \cdot \mathbf{b} = 0$ tell you?
- 2 Which integration method suits $\int x \cos x \, dx$?
- 3 The conjugate of $3 - 2i$?
- 4 What locus is $|z - a| = r$?



Answers 1. they are perpendicular 2. by parts 3. $3 + 2i$ 4. a circle, centre a , radius r