Course Companion

A-Level Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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Pure Mathematics 1

A-Level Mathematics

This handout covers Topic 1: **Pure Mathematics** 纯数学 1. It is the algebra 代数 and calculus 微积分 core of the course. Each ## section is one syllabus subtopic.

Quadratics



Under an idealised uniform deck load, a suspension-bridge cable is approximately parabolic.

A **quadratic** 二次式 is an expression of the form $ax^2 + bx + c$, where $a \neq 0$. The letters a, b, c are the **coefficients** 系数 (the fixed numbers). Much of this section is about solving the **equation** $ax^2 + bx + c = 0$.

■ Completing the square

To **complete the square** 配方 means to write the quadratic in the form

$$a(x + p)^2 + q.$$

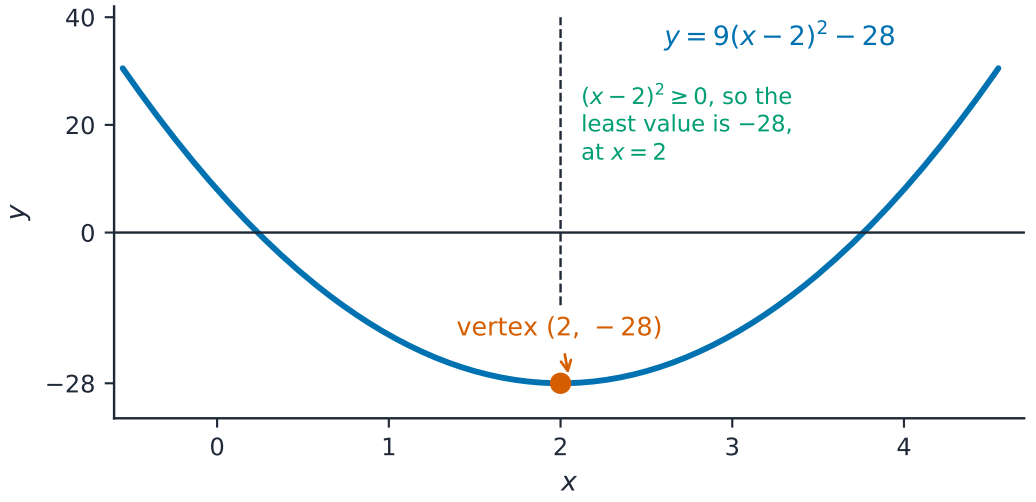
This form is useful: it shows the **vertex** (turning point) of the curve at $(-p,\ q)$, and it gives a quick way to solve the equation.

Worked example. Write $9x^2 - 36x + 8$ in the form $p(x + q)^2 + r$.

Take the factor 9 out of the first two terms 项, then complete the square inside:

$$\begin{aligned} 9x^2 - 36x + 8 &= 9\left(x^2 - 4x\right) + 8 \\ &= 9\left((x-2)^2 - 4\right) + 8 \\ &= 9(x-2)^2 - 36 + 8 = 9(x-2)^2 - 28. \end{aligned}$$

So $p = 9$, $q = -2$, $r = -28$.



$y = a(x-h)^2 + k$ has vertex (h, k) · min if $a > 0$, max if $a < 0$

The completed square hands you the vertex: least value -28 at x equals 2

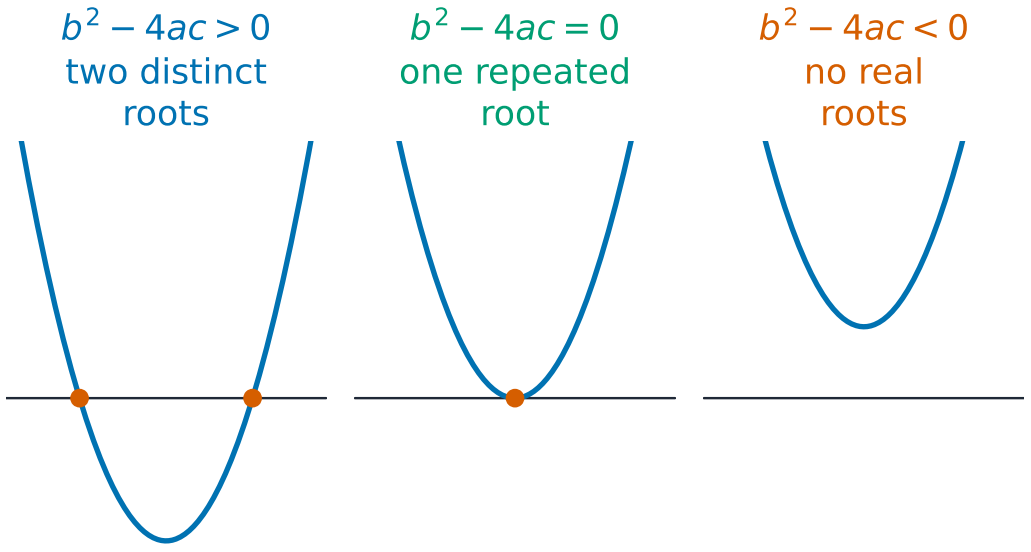
■ The discriminant 判别式

The **discriminant** of $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

It tells you how many **real roots** 实根 (real solutions) the equation $ax^2 + bx + c = 0$ has:

Discriminant	Roots
$b^2 - 4ac > 0$	two **distinct** 相异 real roots
$b^2 - 4ac = 0$	one repeated real root
$b^2 - 4ac < 0$	no real roots



The sign of $b^2 - 4ac$ decides how many times the parabola 抛物线 meets the x-axis.

Worked example. Find the values of the constant k for which $3kx^2+(k+8)x+3 = 0$ has two distinct real roots.

Here $a = 3k$, $b = k + 8$, $c = 3$. For two distinct real roots you need $b^2 - 4ac > 0$:

$$(k + 8)^2 - 4(3k)(3) > 0 \;\Rightarrow\; k^2 + 16k + 64 - 36k > 0 \;\Rightarrow\; k^2 - 20k + 64 > 0.$$

Factorise: $(k - 4)(k - 16) > 0$, so $k < 4$ or $k > 16$. You also need $a \neq 0$, so $k \neq 0$.

■ Quadratic equations 二次方程 and inequalities

To solve a **quadratic equation**, factorise, complete the square, or use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To solve a **quadratic inequality** 二次不等式 such as $(k - 4)(k - 16) > 0$, find the two roots, then decide which side of each root makes the statement true. A sketch of the parabola helps: an upward-opening curve with two distinct roots is above the x-axis (positive) outside the roots and below it (negative) between them.

Worked inequality. Solve $-x^2 + 4x - 3 \geq 0$. Multiplying by -1 reverses the inequality:

$$x^2 - 4x + 3 \leq 0 \quad \Rightarrow \quad (x - 1)(x - 3) \leq 0.$$

The factors have opposite signs between the roots, so $1 \leq x \leq 3$. Equality is allowed at both endpoints. Testing $x = 2$ in the original expression gives -1, which satisfies the inequality; a point outside, such as $x = 0$, gives 3, which does not. Do not automatically choose the outside intervals when the original leading coefficient is negative.

For a negative leading coefficient, the completed-square constant is a maximum: $-2x^2+12x-11=-2(x-3)^2+7$ has maximum 7 at $x=3$. Completing the square still works; the sign changes the interpretation.

■ Simultaneous equations 联立方程

To solve a pair of **simultaneous equations** where one is linear and one is quadratic, use **substitution** 代入: rearrange the linear equation for one letter, then put that into the quadratic. This gives a single quadratic to solve.

Worked substitution. Solve $x+2y=7$ and $x^2+4y^2=25$. The linear equation gives $x=7-2y$, so

$$(7-2y)^2+4y^2=25 \quad \Rightarrow \quad 2y^2-7y+6=0.$$

Factorising gives $(2y-3)(y-2)=0$, hence $y=\frac{3}{2}$ or 2. Recover the matching values of x: the solutions are $(4,\frac{3}{2})$ and $(3,2)$. Each pair gives $x+2y=7$ and $x^2+4y^2=25$. Keep the pairing when presenting answers.

When finding intersections, form the common equation first. For $y=x^2-2x+3$ and $y=x+k$, the intersection equation is $x^2-3x+3-k=0$ and its discriminant is $4k-3$. It has no real intersection for $k<\frac{3}{4}$, a repeated intersection for $k=\frac{3}{4}$, and two for $k>\frac{3}{4}$. The boundary point is $(\frac{3}{2},\frac{9}{4})$.

■ Equations that are quadratic in disguise

Some equations are quadratic in **some function 函数 of** x. For example $x^4-5x^2+4=0$ is quadratic in x^2: let $u=x^2$, solve $u^2-5u+4=0$, then go back to x. You will use this idea again in trigonometry.

Here $(u-1)(u-4)=0$, so $x^2=1$ or 4 and $x=\pm1,\pm2$. Always return from the substitute to the original variable.

Worked square-root substitution. For $6x+\sqrt{x}-1=0$, let $v=\sqrt{x}\geq 0$. Then

$$6v^2+v-1=(3v-1)(2v+1)=0.$$

Only $v=\frac{1}{3}$ is allowed; $v=-\frac{1}{2}$ cannot be a square root. Thus $x=v^2=\frac{1}{9}$, which satisfies the original equation. Squaring a rejected negative value is not a valid way to recover another solution.

■ Using given roots to find constants

A root is an input giving zero. Substitute each stated root into the original polynomial to obtain simultaneous conditions, or expand its root form and compare coefficients. If the roots are α and β, then

$$a(x-\alpha)(x-\beta)=ax^2-a(\alpha+\beta)x+a\alpha\beta.$$

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Worked example. Suppose $px^2 + 6x - 8$ has roots $2m$ and $-4m$, with $p, m \neq 0$. Expanding $p(x-2m)(x+4m)$ gives $px^2+2pmx-8pm^2$. Hence $pm = 3$ and $pm^2 = 1$, giving $m = \frac{1}{3}$ and $p = 9$. Both actual roots $\frac{2}{3}$ and $-\frac{4}{3}$ satisfy $9x^2 + 6x - 8 = 0$. Do not assume a parameter is positive unless the question states that condition.

Functions

A **function** is a rule that sends each input to exactly one output. Write it as $f(x)$. The **composition of functions** 复合函数 $fg(x)$ means apply g first, then apply f to the result.

- The **domain** 定义域 is the set of allowed inputs x.
- The **range** 值域 is the set of outputs the function actually produces.

A function is **one-one** 一一对应 if different inputs always give different outputs. (No output is repeated.) Only a one-one function has an **inverse function** 反函数 f^{-1}, which reverses the rule.

The **composition** 复合 of two functions means doing one after the other. $fg(x)$ means "do g first, then f": $fg(x) = f(g(x))$. The composite function fg exists only when the range of g lies inside the domain of f.

This condition applies to the whole stated domain of the inner function. If $f(x) = \sqrt{x}$ for $x \geq 0$ and $g(x) = x - 3$ for all real x, $fg(x) = \sqrt{x-3}$ is not defined on all of that inner domain. A new restriction $x \geq 3$ makes the formula valid there; state that change explicitly.

To find a range, check whether a boundary is attained. For $x \geq 1$, the range of $1/x$ is $0 < y \leq 1$; zero is approached but never attained. A restricted quadratic also needs its actual domain: $(x-1)^2 + 2$ for $x \geq 3$ has range $y \geq 6$, since its vertex input is excluded.

An open-endpoint condition. Let $F(x) = 1/(x-2)$ for $x > 2$ and $G(x) = 3x + 2$ for $x > a$. The range of G is $(3a + 2, \infty)$. It lies wholly in the domain of F when $3a + 2 \geq 2$, so $a \geq 0$. Equality is allowed because the lower endpoint of that range is not attained. For $a < 0$, the allowed input 0 gives the forbidden output $G(0) = 2$.

■ Finding an inverse

To find f^{-1}: write $y = f(x)$, make x the subject, then swap letters.

Worked example. The function $f(x) = (x+3)^2 - 12$ is defined for $x \geqslant 0$. Find $f^{-1}(x)$.

Write $y = (x+3)^2 - 12$ and solve for x:

$$(x+3)^2 = y + 12 \;\Rightarrow\; x + 3 = \sqrt{y+12} \;\Rightarrow\; x = \sqrt{y+12} - 3.$$

You take the **positive** square root because $x \geqslant 0$ means $x + 3 \geqslant 3 > 0$. So

$$f^{-1}(x) = \sqrt{x+12} - 3.$$

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The original range is $[-3, \infty)$, attained from original inputs $x \geq 0$. Therefore the inverse has domain $x \geq -3$ and range $f^{-1}(x) \geq 0$. Merely requiring the radical to be real would give the larger, incorrect inverse domain $x \geq -12$.

A different restriction can require the negative square-root branch. For $h(x) = (x + 1)^2 - 5$ with $x \leq -1$, solve $y + 5 = (x + 1)^2$ using $x + 1 \leq 0$:

$$h^{-1}(x) = -1 - \sqrt{x + 5}, \qquad x \geq -5.$$

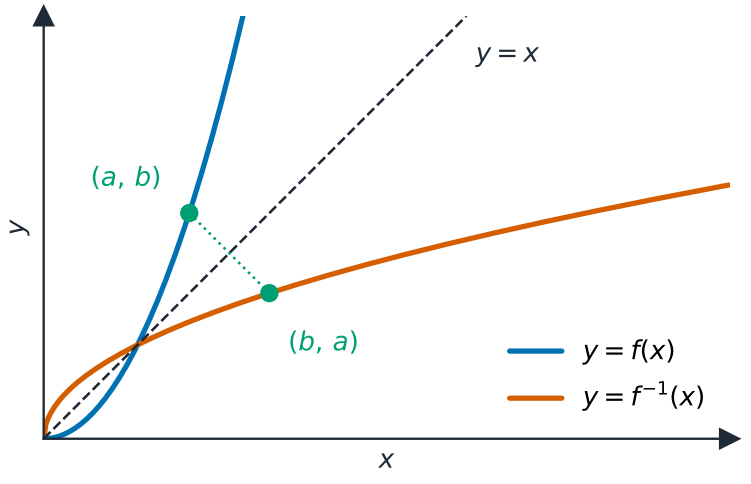
Its range is $h^{-1}(x) \leq -1$. The endpoint $(-1, -5)$ reflects to $(-5, -1)$ and the point $(-3, -1)$ reflects to $(-1, -3)$. A $\pm$ answer would not define a one-one inverse function. The inverse is also different from the reciprocal $1/h(x)$.

To invert a composite of compatible one-one functions, reverse their order: $(gf)^{-1} = f^{-1}g^{-1}$. Alternatively form the composite rule first, then invert it with the original domain restriction.

For example, $f(x) = (x - 1)^2 + 2$ for $x \leq 1$ and $g(x) = 5 - x$ give $H(x) = gf(x) = 3 - (x - 1)^2$, with range $H(x) \leq 3$. Solving $y = 3 - (x - 1)^2$ using $x - 1 \leq 0$ gives $H^{-1}(x) = 1 - \sqrt{3 - x}$ for $x \leq 3$, with range $H^{-1}(x) \leq 1$. Composing and then inverting keeps the whole branch condition visible.

■ Graphs of inverses and transformations

The graph of $y = f^{-1}(x)$ is the **reflection** of $y = f(x)$ in the line $y = x$.



inverse = reflection in $y = x$ · domain of f^{-1} = range of f

Reflecting $y = f(x)$ in the line $y = x$ gives its inverse; a point (a, b) becomes (b, a).

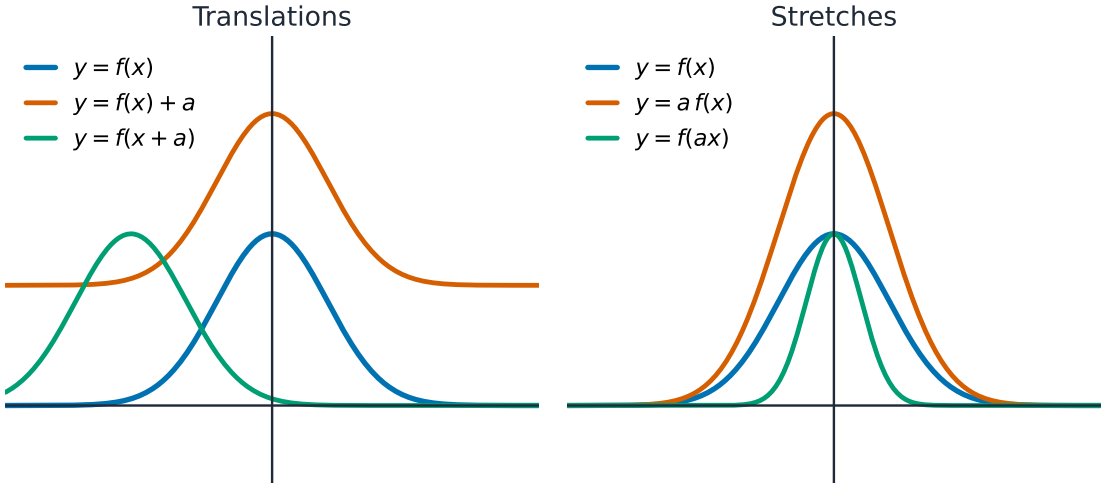
You should know these **transformations** 变换 of $y = f(x)$: In the stretch descriptions below take $a > 0$. Negative multipliers also require a reflection 反射: $-f(x)$ reflects in the x-axis and $f(-x)$ reflects in the y-axis.

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New equation	Effect on the graph
$y = f(x) + a$	**translation** 平移 up by a
$y = f(x + a)$	translation left by a
$y = a\,f(x)$	**stretch** in the y-direction, scale factor a
$y = f(ax)$	stretch in the x-direction, scale factor $\frac{1}{a}$

When two transformations are combined, the order can matter. State each one fully (type, direction, and amount).

For $g(x) = 2f(x+3) - 1$, use two transformations: stretch vertically by factor 2, then translate by $(-3, -1)$ (left 3 and down 1 together). A point $(4, 5)$ becomes $(4, 10)$, then $(1, 9)$. Reversing these same operations gives $2(f(x + 3) - 1) = 2f(x + 3) - 2$, a different graph. If two transformations are required, combine the horizontal and vertical shifts into one translation. For a horizontal multiplier $f(ax)$, a point (s, t) moves to $(s/a, t)$; the scale factor is reciprocal, not a.



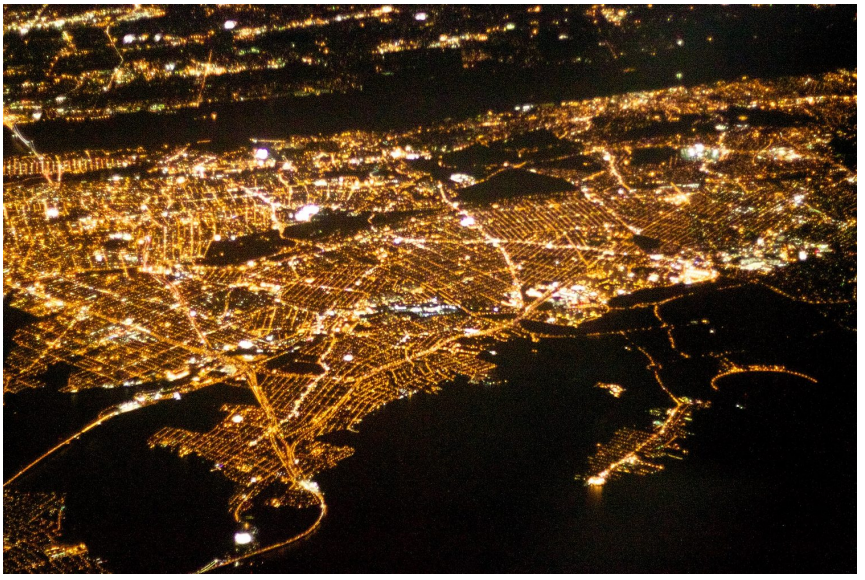
$f(x) + a$ moves up · $f(x + a)$ moves left · $af(x)$ stretch y · $f(ax)$ compress x if $a > 1$

Adding to the output or input slides the curve; a multiplier stretches it.

Coordinate geometry

Coordinate geometry 坐标几何 connects positions, distances and intersections with algebra. A straight path can cross a circular boundary twice, touch it once or miss it. Solve their equations together to distinguish these cases exactly.

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A city street grid: like coordinates, every point is fixed by two numbers measured from a pair of axes

■ Choose and check a line equation

For two points with different x-coordinates, the **gradient** 斜率 is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Useful forms of an **equation of a straight line** 直线方程 are

$$y = mx + c, \qquad y - y_1 = m(x - x_1), \qquad ax + by + c = 0.$$

For example, through $(2, 3)$ and $(6, 11)$ the gradient is 2. The same line is $y - 3 = 2(x - 2)$, $y = 2x - 1$, or $2x - y - 1 = 0$. Substitute each given point to check. In $2x - 3y + 6 = 0$, rearranging gives $y = \frac{2}{3}x + 2$, so the gradient is $2/3$, not 2.

Distinct non-vertical **parallel** 平行 lines have equal gradients. For **perpendicular** 垂直 lines with finite gradients, $m_1 m_2 = -1$. Horizontal lines $y = h$ have gradient zero; vertical lines $x = k$ have undefined gradient. They are perpendicular, but the product rule cannot be used for a vertical line. Through $(4, -2)$, the horizontal and vertical lines are $y = -2$ and $x = 4$.

■ Distances, midpoints 中点 and perpendicular bisectors 垂直平分线

Pythagoras gives the distance between two points:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The **midpoint** is

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$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

A **perpendicular bisector** passes through this midpoint at right angles to the segment. Every point on it is equally far from the two ends.

> **Worked example.** For $P(-1, 2)$ and $Q(5, 6)$, $PQ = \sqrt{6^2 + 4^2} = 2\sqrt{13}$ and $M = (2, 4)$. The segment gradient is $2/3$, so the bisector gradient is $-3/2$. Its equation is
>
> $$y - 4 = -\frac{3}{2}(x - 2) \quad \Longrightarrow \quad 3x + 2y = 14.$$

The midpoint alone is not an equation, and a perpendicular through an endpoint is not a bisector. If PQ is horizontal, its bisector is the vertical line through M; do not take a reciprocal of zero.

■ Read a circle 圆 in two forms

A **circle** with **centre** 圆心 (a, b) and positive **radius** 半径 r has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

An expanded equation needs both squares completed:

$$\begin{aligned} x^2 + y^2 - 6x + 10y - 27 &= 0, \\ (x - 3)^2 - 9 + (y + 5)^2 - 25 &= 27, \\ (x - 3)^2 + (y + 5)^2 &= 61. \end{aligned}$$

The centre is $(3, -5)$ and radius $\sqrt{61}$. More generally, $x^2 + y^2 + 2gx + 2fy + c = 0$ gives centre $(-g, -f)$ and $r^2 = g^2 + f^2 - c$. Check that $r^2 > 0$ for a real circle of positive radius. If it is negative there are no real points; if zero there is only one point. Substitute a proposed point into the equation to test it.

■ Use a diameter 直径, a semicircle or symmetry

> **Worked example.** The points $P(1, 1)$ and $Q(7, 11)$ are the ends of a **diameter**. Its midpoint is $(4, 6)$ and its half-length is
>
> $$r = \frac{1}{2}\sqrt{6^2 + 10^2} = \sqrt{34}.$$

The circle is $(x - 4)^2 + (y - 6)^2 = 34$. A general chord's 弦 midpoint is not necessarily the centre: the diameter condition matters.

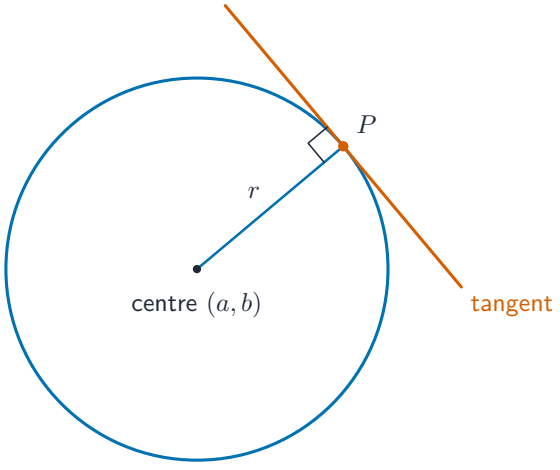
The angle formed at any other point on this circle by lines to P and Q is $90°$: it is an angle in a semicircle. For a non-diameter chord, the line from the centre to its midpoint is perpendicular to the chord and is a symmetry line. Use this to recover a reflected intersection point or a missing coordinate.

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For instance, $(x-2)^2+(y-1)^2=5$ has diameter ends $A(0,0)$ and $B(4,2)$. Point $D(4,0)$ lies on it and $\angle ADB=90°$. The chord AD has midpoint $(2,0)$, so its perpendicular bisector $x=2$ passes through the centre and reflects A onto D.

■ Tangents 切线: known and unknown contact points

A **tangent** is perpendicular to the radius at the contact point.



A tangent meets the radius at a right angle.

For the circle centred at $(2,1)$ above, the radius to $B(4,2)$ has gradient $1/2$. The tangent has gradient -2, so $y-2=-2(x-4)$, or $2x+y=10$. Use B in this equation, not the centre. At the top and bottom the tangents are horizontal, $y=1\pm\sqrt{5}$; at the left and right they are vertical, $x=2\pm\sqrt{5}$.

> **Worked example: unknown contact points.** Find both points on this circle where the tangent gradient is -2. The radius gradient must be $1/2$, and the radius line through the centre is $y-1=\frac{1}{2}(x-2)$. Put $u=x-2$ and substitute:

$$u^2+\frac{u^2}{4}=5 \quad\Longrightarrow\quad u=\pm 2.$$

The contact points are $(4,2)$ and $(0,0)$, with tangent equations $2x+y=10$ and $2x+y=0$. A given tangent direction normally gives two opposite contact points. Keep exact square roots when the solutions are not integers. Radius geometry and substitution suffice; circle differentiation 微分 is not required here.

■ Intersections and parameter boundaries

Substitute a line into a circle to get a quadratic, solve for one coordinate and recover the other. In $(x-2)^2+(y-1)^2=5$, the line $y=0$ gives $(x-2)^2=4$, hence the points $(0,0)$ and $(4,0)$.

For the family $y=k$, the same circle gives

$$(x-2)^2 = 5-(k-1)^2.$$

There are two points when $1-\sqrt{5} < k < 1+\sqrt{5}$, one tangent point at each boundary, and no points outside that interval. At either boundary $x = 2$. For a general sloping line, the resulting quadratic's discriminant is positive, zero or negative in the same three cases. A repeated root is one point.

For a line and a quadratic curve, equate the two expressions for y. For $y = x^2$ and $y = 2x + k$:

$$x^2 - 2x - k = 0, \qquad \Delta = 4 + 4k.$$

Thus there are two intersections for $k > -1$, one for $k = -1$ at $(1, 1)$, and none for $k < -1$. Show the algebraic condition; a sketch alone does not prove an exact boundary. Some families have a positive discriminant for every real parameter, so they always intersect twice. Check separately if a parameter could make the quadratic coefficient zero.

■ Build a circle from axis gaps

A circle entirely in the first quadrant has centre (a, b) and radius r. Its least distances from the y-axis and x-axis are $a - r$ and $b - r$, respectively.

> **Worked example.** The gap from the y-axis is 1, the gap from the x-axis is 3, and the distance from the origin to the centre is $\sqrt{34}$. Then the centre is $(r + 1, r + 3)$, so

$$\begin{aligned}(r+1)^2 + (r+3)^2 &= 34,\\ r^2 + 4r - 12 &= 0,\\ (r+6)(r-2) &= 0.\end{aligned}$$

Reject the negative radius. Therefore $r = 2$, centre $(3, 5)$, and equation $(x-3)^2+(y-5)^2 = 4$. The point furthest from the origin lies along the ray from the origin through the centre: its radius has gradient $5/3$, so its tangent gradient is $-3/5$.

Independent check. A different circle's least gaps from the x- and y-axes are 6 and 2, and its centre is $4\sqrt{5}$ from the origin. Find the radius, centre, circle equation and tangent gradient at its furthest point from the origin. Set up the centre coordinates before calculating.

Check after attempting: centre $(r + 2, r + 6)$ gives $(r + 2)^2 + (r + 6)^2 = 80$. Thus $(r + 10)(r - 2) = 0$, $r = 2$, centre $(4, 8)$ and equation $(x - 4)^2 + (y - 8)^2 = 4$. The radius direction has gradient 2, so the tangent gradient is $-1/2$.

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Circular measure

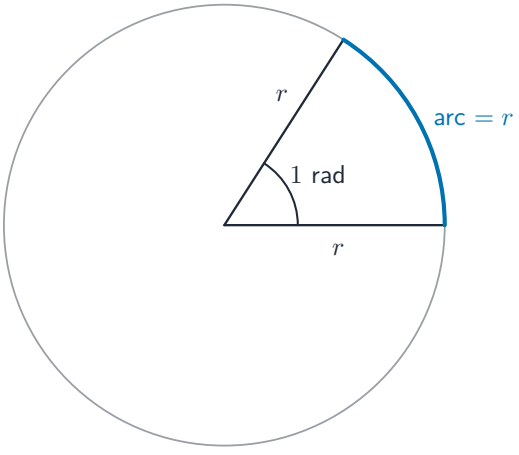
A curved boundary may contain an arc 弧 and several straight edges. Identify every piece before calculating. A sector 扇形, a triangle and a circular segment 弓形 have different areas and perimeters.

■ Define a radian 弧度 and convert angles

One **radian** is the angle at the centre that cuts off an **arc** equal in length to the radius. Since $\theta = s/r$, a full turn is $(2\pi r)/r = 2\pi$ radians. Therefore

$$\pi \text{ radians} = 180^\circ.$$

Multiply degrees 度 by $\pi/180$ to get radians, or radians by $180/\pi$ to get degrees. For example, $150^\circ = 5\pi/6$ radians and $7\pi/4$ radians is 315°. Angles can exceed one turn: $450^\circ = 5\pi/2$ radians.



one **radian**: the angle whose arc is exactly one radius long

π rad $= 180^\circ$, so 1 rad $\approx 57.3^\circ$

degrees → radians: $\times \dfrac{\pi}{180}$ radians → degrees: $\times \dfrac{180}{\pi}$

One radian is about 57.3°.

■ Sector formulas, perimeter and reverse problems

For a **sector** with radius r and angle θ in radians,

$$s = r\theta, \qquad A = \frac{1}{2}r^2\theta.$$

Here s is **arc length** and A is sector area. A sector's full perimeter is $s+2r$. Distinguish linear units from square units. Convert a degree angle before using these formulas, and use radian calculator mode for its sine 正弦 or cosine 余弦.

If the arc length and area are given, eliminate the angle: $A = rs/2$, so $r = 2A/s$ and then $\theta = s/r$.

> **Worked example.** Arc length 8 cm and sector area 24 cm^2 give $r = 6$ cm, $\theta = 4/3$ radians and full perimeter 20 cm. The arc alone is not the perimeter.

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Chords and minor circular segments

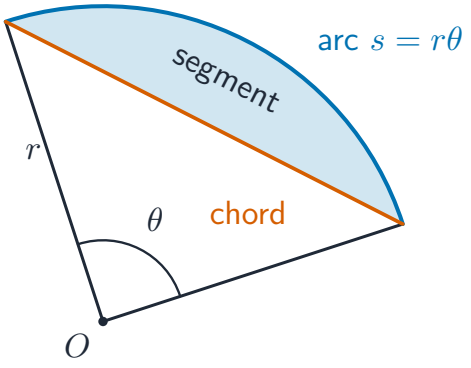
For a minor central angle $0 < \theta < \pi$, bisect the isosceles triangle formed by two radii and their chord. This gives

$$c = 2r\sin(\theta/2), \qquad A_{\text{triangle}} = \frac{1}{2}r^2\sin\theta.$$

The **circular segment** lies between the chord and arc. The minor segment is the minor sector minus the triangle:

$$A_{\text{minor}} = \frac{1}{2}r^2(\theta - \sin\theta).$$

Its perimeter is $r\theta + c$, not $r\theta + 2r$.



The chord separates the triangle from the minor segment.

For $r = 5$ cm and $\theta = 2$ radians, the arc length is 10 cm, the sector area is 25 cm^2, and the minor segment area is $12.5(2 - \sin 2) \approx 13.6$ cm^2. The minor segment perimeter is $10 + 10\sin 1 \approx 18.4$ cm.

Major segments

The major segment is the rest of the circle after removing the minor segment:

$$A_{\text{major}} = \pi r^2 - A_{\text{minor}}.$$

Its arc uses angle $2\pi - \theta$, so its perimeter is $r(2\pi - \theta) + c$.

Worked example. Radius 4 cm and chord $4\sqrt{3}$ cm give $\sin(\theta/2) = \sqrt{3}/2$. The minor angle is $2\pi/3$, the triangle area is $4\sqrt{3}$, and

$$A_{\text{minor}} = \frac{16\pi}{3} - 4\sqrt{3},$$
$$A_{\text{major}} = \frac{32\pi}{3} + 4\sqrt{3}.$$

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The major arc length is $16\pi/3$ cm. Add the chord for its segment perimeter. The major region 区域 includes the triangle; it is not just the major sector.

■ Compound boundaries: name the pieces

Worked example. Let $OA = OB = 10$ cm, $\angle AOB = 1$ radian. Remove the sector of radius 4 cm, centred at O, from triangle OAB. The remaining boundary contains two straight gaps of length 6, the outer chord $AB = 20\sin(1/2)$, and an inner arc of length 4.

$$P = 6 + 6 + 20\sin(1/2) + 4 \approx 25.6 \text{ cm},$$
$$A = \frac{1}{2}(10)^2 \sin 1 - \frac{1}{2}(4)^2(1) \approx 34.1 \text{ cm}^2.$$

There is no outer circular arc: AB is straight. If instead a region lies between two concentric sector arcs, its area is the difference of sector areas, but its perimeter includes both arcs and both radial gaps. For radii 9 and 5 and angle 1.4 radians, the area is 39.2 square units and perimeter is 27.6 units. Subtracting areas does not mean subtracting lengths.

■ Use two conditions to find an unknown circle

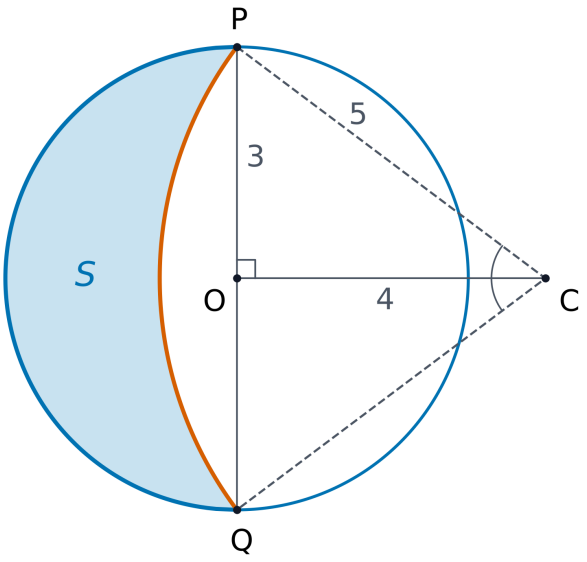
If a sector has area 8θ cm^2, then $(1/2)r^2\theta = 8\theta$. Since $\theta > 0$, $r = 4$ cm. If the triangle area is 4 cm^2, then $8\sin\theta = 4$. The restriction $0 < \theta < \pi/2$ selects $\theta = \pi/6$, giving minor segment area $4\pi/3 - 4$ cm^2.

If a later part replaces the sector-area condition, start again. A chord-to-radius ratio can determine θ using $c = 2r\sin(\theta/2)$, then triangle area determines r. Do not keep a radius or angle whose original condition no longer applies.

■ Different circles, different radii and angles

The diagram shows a circle with centre O and radius 3, meeting a circle with centre C and radius 5. Here $OC = 4$ and PQ is the small circle's diameter, perpendicular to OC.

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Only the relevant arc of the large circle is shown. Region S is inside the small circle and outside the large circle.

Let $\phi = \angle PCQ = 2\tan^{-1}(3/4) \approx 1.287$ radians. Triangle PCQ has area $\frac{1}{2}(6)(4) = 12$. The large circle's minor segment has area $\frac{25}{2}\phi - 12$. Subtract it from the small left semicircle:

$$A_S = \frac{9}{2}\pi - \frac{25}{2}\phi + 12 \approx 10.0,$$
$$P_S = 3\pi + 5\phi \approx 15.9.$$

The perimeter uses two curved pieces. The lines PQ and OC are construction lines inside the picture, not outside edges.

A shared chord can split another two-circle region into two circular segments. Sketch the required side of the chord for each circle. Calculate each sector and triangle with its own radius and central angle before combining their areas. For example, two circles of radius 6, with centres 6 apart, overlap in two equal minor segments. Equilateral triangles give each central angle $2\pi/3$. The overlap area is $2(12\pi - 9\sqrt{3}) = 24\pi - 18\sqrt{3}$ and its perimeter is the two minor arcs, 8π. The shared chord is internal.

Independent check. Replace the small radius by 4, set $OC = 3$, and keep the larger radius 5, with the same perpendicular-diameter geometry. Find the crescent area and perimeter to three significant figures.

Check after attempting: $\phi = 2\tan^{-1}(4/3)$. The large triangle still has area 12. Thus area $= 8\pi - \frac{25}{2}\phi + 12 \approx 14.0$ square units and perimeter $= 4\pi + 5\phi \approx 21.8$ units.

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Trigonometry

A point on a turning wheel rises and falls. A trigonometric model describes that cycle, but its graph must have the correct height scale and timing. One calculator angle may miss another valid position in the same interval.



A point on a turning wheel can be modelled with a sine or cosine function.

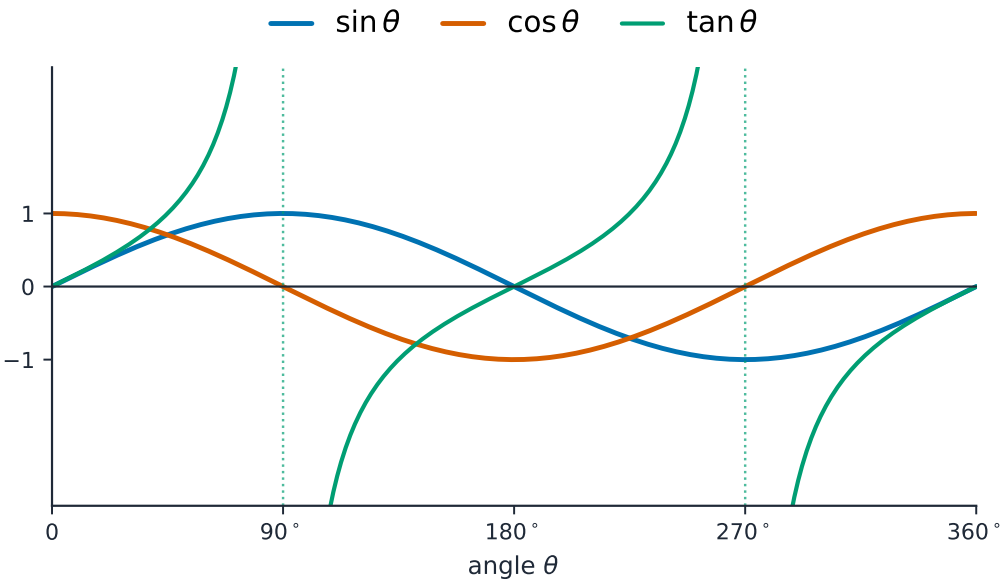
■ Exact values and signs

On the unit circle a point has coordinates $(\cos\theta, \sin\theta)$. Signs follow the quadrant, including negative angles and turns beyond 360°. The sine and cosine graphs have range $[-1, 1]$ and period 周期 2π radians or 360°. The tangent function 正切 repeats after π radians or 180° and is undefined where $\cos\theta = 0$.

θ	30°	45°	60°
$\sin\theta$	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$
$\cos\theta$	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$
$\tan\theta$	$1/\sqrt{3}$	1	$\sqrt{3}$

Use the **reference angle** 参考角, the acute angle to the x-axis, to get the size. Then choose the sign. For example, $\cos 150° = -\sqrt{3}/2$, $\sin(-\pi/6) = -1/2$ and $\sin(3\pi/4) = 1/\sqrt{2}$. The notation $\sin^2\theta$ means $(\sin\theta)^2$.

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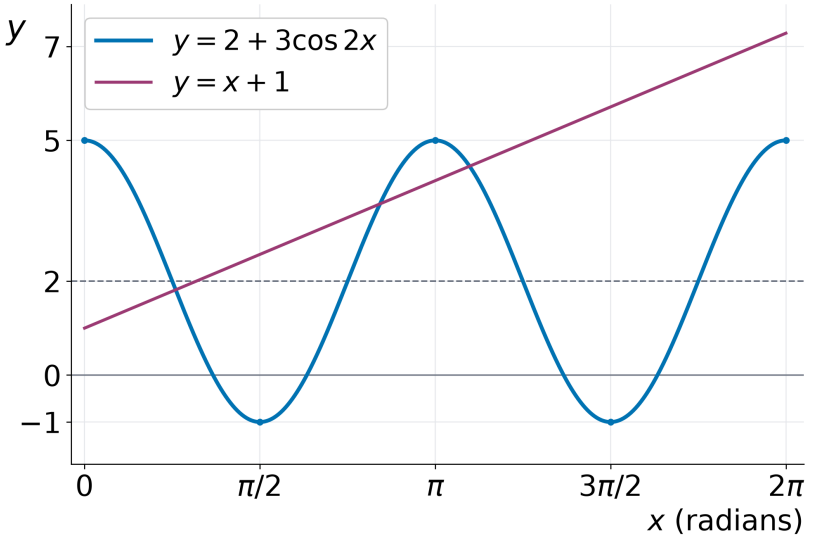


The tangent branches stay separate at their vertical asymptotes.

■ Transformed graphs

For $y = d + a\sin(bx + c)$ or $d + a\cos(bx + c)$, with $a, b \neq 0$, the midline is $y = d$, **amplitude** 振幅 is $|a|$, and **period** is $2\pi/|b|$ radians or $360°/|b|$. Find key points by setting the angle inside the function to familiar values.

For $y = 2 + 3\cos 2x$, the range is $[-1, 5]$ and period is π. On $[0, 2\pi]$, maxima occur at $x = 0, \pi, 2\pi$ and minima at $x = \pi/2, 3\pi/2$. There are two complete cycles.



The comparison line $y = x + 1$ meets this curve three times. This counts the solutions of $2 + 3\cos 2x = x + 1$ on the displayed interval.

For $y = \tan(x + \pi/4)$, shift the base tangent graph left by $\pi/4$. In $[-\pi, \pi]$, asymptotes are $x = -3\pi/4, \pi/4$, zeros are $-\pi/4, 3\pi/4$ and the y-intercept is 1. Its period is π and it has no finite amplitude. Draw separate increasing branches; never join through an asymptote.

■ Principal values 主值 and inverse notation

The notations $\sin^{-1} u$, $\cos^{-1} u$ and $\tan^{-1} u$ return a **principal value**, an angle in the stated inverse range:

Inverse	Principal range in radians	In degrees
Sine	$[-\pi/2, \pi/2]$	$[-90°, 90°]$
Cosine	$[0, \pi]$	$[0°, 180°]$
Tangent	$(-\pi/2, \pi/2)$	$(-90°, 90°)$

Sine/cosine inverse require an input in $[-1, 1]$; tangent inverse accepts every real input. The superscript -1 means inverse here, not reciprocal. Thus $\sin^{-1}(\sin 150°) = 30°$, but $\sin\theta = 1/2$ has both $30°$ and $150°$ in $[0°, 360°]$.

To solve $\tan^{-1}(2x - 1) = -\pi/4$, take tangent: $2x - 1 = -1$, so $x = 0$. The given inverse output is inside the principal range.

■ Prove identities with valid algebra

The two core **identities** are

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}, \qquad \sin^2\theta + \cos^2\theta \equiv 1.$$

An identity 恒等式 holds wherever both sides are defined. State denominator restrictions, and start from one side rather than assuming the equality you must prove. For example, where $\sin\theta \neq 0$:

$$\begin{aligned}\frac{1-\cos\theta}{\sin\theta} &= \frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)} \\ &= \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} \\ &= \frac{\sin\theta}{1+\cos\theta}.\end{aligned}$$

The condition also excludes $\cos\theta = -1$, so the cancelled factor is non-zero. Do not confuse $\sin^2\theta + \cos^2\theta$ with $(\sin\theta + \cos\theta)^2$.

A trigonometric equation 三角方程 asks for all angles in its stated interval.

Reduce equations to one trig function

Worked example. Solve $2\cos^2\theta - \sin\theta - 1 = 0$ on $[0, 2\pi]$. Substitute $\cos^2\theta = 1 - \sin^2\theta$:

$$2\sin^2\theta + \sin\theta - 1 = 0 \quad \Longrightarrow \quad (2\sin\theta - 1)(\sin\theta + 1) = 0.$$

Therefore $\sin\theta = 1/2$ or -1. The full answer is $\theta = \pi/6, 5\pi/6, 3\pi/2$. The principal value $-\pi/2$ is not in this interval, so use $3\pi/2$ instead. Reject sine/cosine values outside $[-1, 1]$. If solving for $\sin^2\theta$, recover both square-root signs before finding the angles.

Change the inner interval

For $\sin 2x = 1/2$ and $-\pi < x < \pi$, let $u = 2x$. Search the whole interval $-2\pi < u < 2\pi$, not just one turn. Its solutions are $u = -11\pi/6, -7\pi/6, \pi/6, 5\pi/6$. Dividing by 2 gives

$$x = -11\pi/12, -7\pi/12, \pi/12, 5\pi/12.$$

For a shifted angle such as $2x - \pi/6$, transform both interval endpoints first, then convert every inner-angle solution back to x. Check open/closed endpoints in the original equation.

Keep zero factors and exclude undefined angles

Dividing by a trig expression can lose roots. Write $\sin\theta\cos\theta = \sin\theta$ as $\sin\theta(\cos\theta - 1) = 0$, retaining the case $\sin\theta = 0$.

For a fractional equation, first state the denominator restriction. For example, solve $6\sin\theta = 1 + 2/\sin\theta$ for $-180° < \theta < 180°$. Since $\sin\theta \neq 0$, multiplication gives

$$(3\sin\theta - 2)(2\sin\theta + 1) = 0.$$

The value $2/3$ gives $41.8°, 138.2°$ and $-1/2$ gives $-150°, -30°$. All four are valid in the original equation. Keep calculator precision until the final angles.

Independent graph and equation checks

Sketch $y = 1 + \sin x$ and $y = x/2$ together for $0 \leq x \leq 2\pi$. Label the sine curve's five key points and count solutions of $1 + \sin x = x/2$.

Check after attempting: the curve passes through $(0, 1), (\pi/2, 2), (\pi, 1), (3\pi/2, 0), (2\pi, 1)$. There is one intersection, between $\pi/2$ and π. There the curve falls and the line rises. Before $\pi/2$ the curve is at least 1 and the line below 1; after π the curve is at most 1 and the line above 1.

Also solve $\sin(2x - \pi/6) = -1/2$ on $[-\pi, \pi]$. The inner interval is $[-13\pi/6, 11\pi/6]$, giving $x = -\pi, -\pi/3, 0, 2\pi/3, \pi$. Both endpoints are valid. A correct list of properties

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does not replace an actual graph, and one principal inverse value does not replace all equation solutions.

Series

■ The binomial expansion 二项展开式

For a positive integer n, the **binomial expansion** is

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n,$$

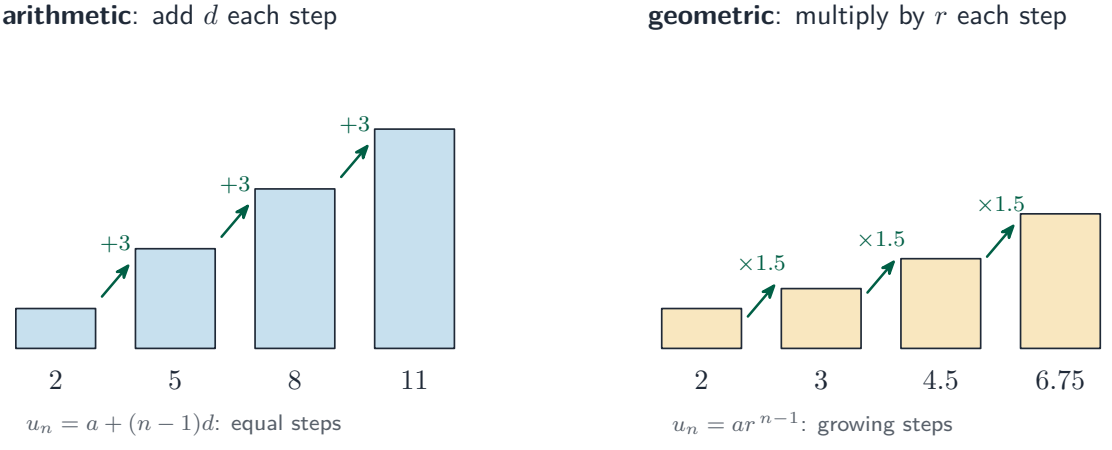
where $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$ is a **binomial coefficient** 二项式系数.

> **Worked example.** Find the first three terms, in ascending powers of x, of $(2-px)^5$.
>
> $$(2-px)^5 = 2^5 + \binom{5}{1}2^4(-px) + \binom{5}{2}2^3(-px)^2 + \cdots = 32 - 80px + 80p^2x^2 + \cdots$$

■ Arithmetic and geometric progressions 等比数列

A **progression** 数列 (sequence) is a list of terms following a rule.

- An **arithmetic progression** 等差数列 (AP) adds a fixed **common difference** 公差 d each step. The nth **term** is $u_n = a + (n-1)d$, and the sum of the first n terms is $S_n = \frac{n}{2}\big(2a + (n-1)d\big)$.
- A **geometric progression** (GP) multiplies by a fixed **common ratio** 公比 r each step. The nth term is $u_n = ar^{\,n-1}$, and $S_n = \dfrac{a(1-r^n)}{1-r}$.



An AP climbs in equal steps; a GP's steps grow by the same ratio

For a nonzero first term, a geometric series is **convergent** – its partial sums **converge**

收敛 (settle to a limit 极限) – exactly when $|r| < 1$. Then it has a **sum to infinity**

$$S_\infty = \frac{a}{1-r}.$$

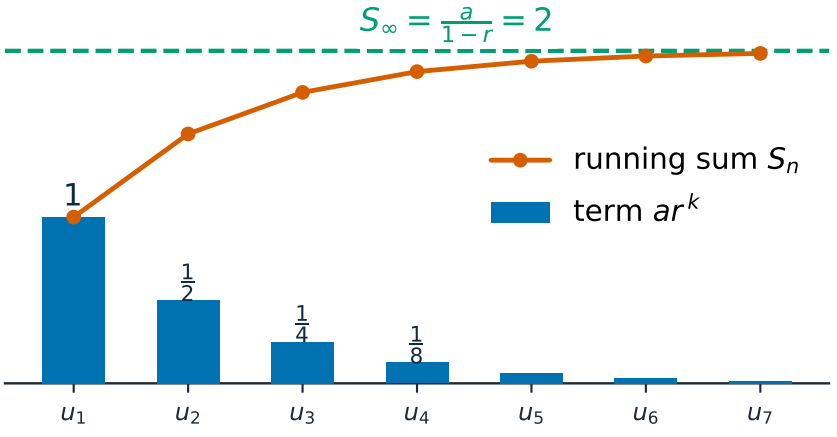
> **Worked example.** The third term of a GP is 18 and the sum of the first three terms is 26. The common ratio is negative. Find the sum to infinity.

From $ar^2 = 18$ you get $a = \dfrac{18}{r^2}$. Put this into $a(1 + r + r^2) = 26$:

$$18(1 + r + r^2) = 26r^2 \;\Rightarrow\; 8r^2 - 18r - 18 = 0 \;\Rightarrow\; (4r + 3)(r - 3) = 0.$$

The ratio is negative, so $r = -\frac{3}{4}$ and $a = \dfrac{18}{(3/4)^2} = 32$. Then

$$S_\infty = \frac{32}{1 - (-\frac{3}{4})} = \frac{32}{\frac{7}{4}} = \frac{128}{7}.$$



$S_n = \frac{a(1-r^n)}{1-r}$, $S_\infty = \frac{a}{1-r}$ only if $|r| < 1$

Convergent GP: bars shrink toward zero while the running sum approaches S infinity = a/(1-r)

■ Select a binomial term or a coefficient in a product

The term numbered $k + 1$ in $(a + b)^n$ is $\binom{n}{k} a^{n-k} b^k$, with $k = 0, \ldots, n$ and $0! = 1$. The power n is a positive integer here. Keep the sign inside the powered second part.

> **Worked example.** In $(x^2 + 2/x)^6$, the general term is $\binom{6}{k} 2^k x^{12-3k}$. The constant term needs $12 - 3k = 0$, so $k = 4$ and the answer is $15 \times 16 = 240$.

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For the coefficient of x^2 in $(1+3x)(2-x)^4$, only two pairings contribute: the constant 1 times $24x^2$, and $3x$ times $-32x$. The coefficient is $24-96=-72$. When negative powers occur, list every possible pairing before selecting the terms.

■ Find an unknown AP and select its terms

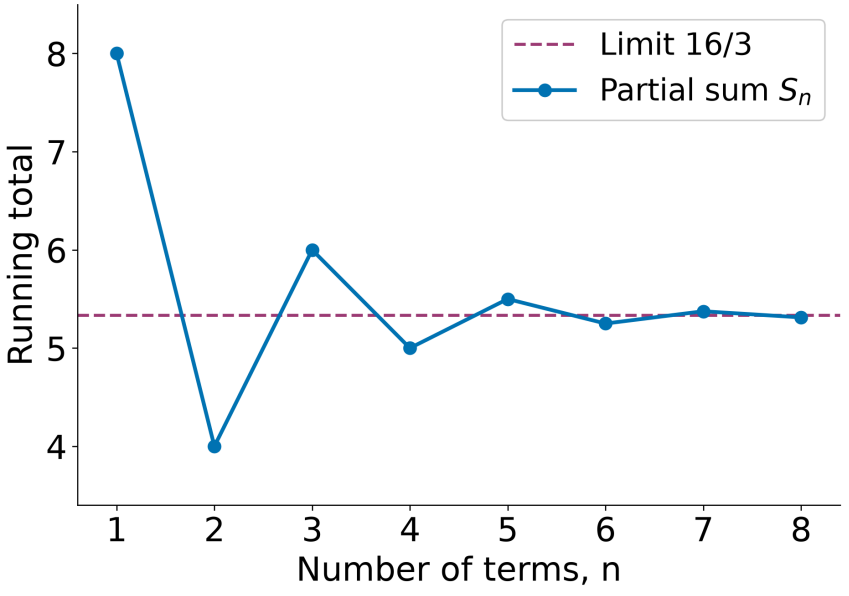
If $u_3=13$ and $u_8=33$, then $a+2d=13$ and $a+7d=33$. Subtraction gives $d=4$ and $a=5$. Hence $u_{10}=41$ and $S_{10}=\frac{10}{2}(5+41)=230$.

For $a=4, d=3$, the terms strictly between 20 and 41 satisfy $20<3n+1<41$. Their positions are 7 through 13: there are $13-7+1=7$ terms, from 22 to 40. The sum is $\frac{7}{2}(22+40)=217$, also $S_{13}-S_6$. Check whether the endpoints are included in the question.

■ Connect progressions and check convergence

Three consecutive AP terms p, q, s satisfy $2q=p+s$. Three consecutive GP terms satisfy $q^2=ps$. If p, q, s are GP terms while $p, q, -2s$ are AP terms, substitute $q=pr, s=pr^2$ into $2q=p-2s$. For $p\neq 0$, this gives $2r^2+2r-1=0$. Solve, then apply any sign or convergence restrictions.

For a finite GP with $r=1$, use $S_n=na$; the fraction formula has a zero denominator. For a non-zero first term, an infinite geometric series has a finite sum exactly when $|r|<1$. A negative ratio can converge: signs alternate and the partial sums approach a limit from both sides. For $r=-1$ they keep oscillating instead.



For $a=8, r=-1/2$, the limiting sum is $16/3$.

After the first n terms are removed, the remaining sum is

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$$S_\infty - S_n = \frac{ar^n}{1-r}.$$

Worked example. A second term of -12 and infinite sum 16 give $ar = -12$ and $a = 16(1-r)$. Thus $4r^2 - 4r - 3 = 0$. Reject $r = 3/2$ because it cannot converge, leaving $r = -1/2, a = 24$. After three terms the remaining infinite sum is $24(-1/2)^3/(3/2) = -2$. A negative tail is valid: the finite total was above the limiting sum.

Differentiation

Differentiation finds the **gradient of a curve** 曲线斜率 at each point. The gradient is the **limit** of the gradients of shorter and shorter **chords**, called the **derivative** 导数.

The rules

Write the derivative as $f'(x)$ or $\dfrac{dy}{dx}$. The basic rule is

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \quad \text{for any rational } n.$$

Differentiate sums term by term, and use the **chain rule** 链式法则 for a function inside a function:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Differentiating again gives the **second derivative** 二阶导数 $f''(x)$ or $\dfrac{d^2y}{dx^2}$.

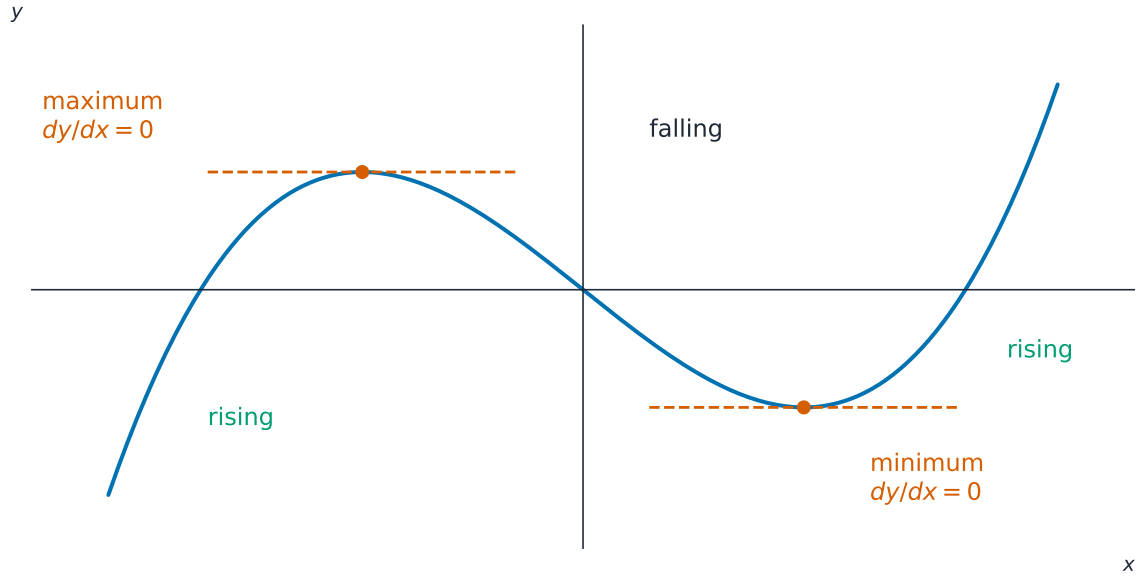
Using the derivative

- **Tangent and normal 法线.** The gradient of the curve at a point is the gradient of the **tangent** there. The **normal** is perpendicular to the tangent, so its gradient is $-\dfrac{1}{(\text{tangent gradient})}$ when the tangent gradient is non-zero. A horizontal tangent has a vertical normal.
- **Increasing or decreasing.** The function is an **increasing function** 增函数 where $\dfrac{dy}{dx} > 0$, and a **decreasing function** 减函数 where $\dfrac{dy}{dx} < 0$.
- **Rate of change 变化率.** A derivative is a **rate of change**. Linked rates use the chain rule, e.g. $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$.

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■ Stationary points 驻点

A **stationary point** is where $\dfrac{dy}{dx} = 0$. Test its nature with the second derivative: $f''(x) > 0$ gives a **minimum point** 极小值点, and $f''(x) < 0$ gives a **maximum point** 极大值点. When $f''(x) = 0$ the test is inconclusive. Check the sign of the first derivative on both sides: negative to positive proves a minimum, and positive to negative proves a maximum. For example, $y = x^4$ has a minimum at zero despite both derivatives being zero there.



At a maximum or a minimum the tangent is flat, so $\frac{dy}{dx} = 0$.

Worked example. The curve $y = 4x^{1/2} - x$ has a maximum point at $x = a$. Find a.

$$\frac{dy}{dx} = 2x^{-1/2} - 1 = 0 \;\Rightarrow\; \frac{2}{\sqrt{x}} = 1 \;\Rightarrow\; \sqrt{x} = 2 \;\Rightarrow\; x = 4.$$

So $a = 4$.

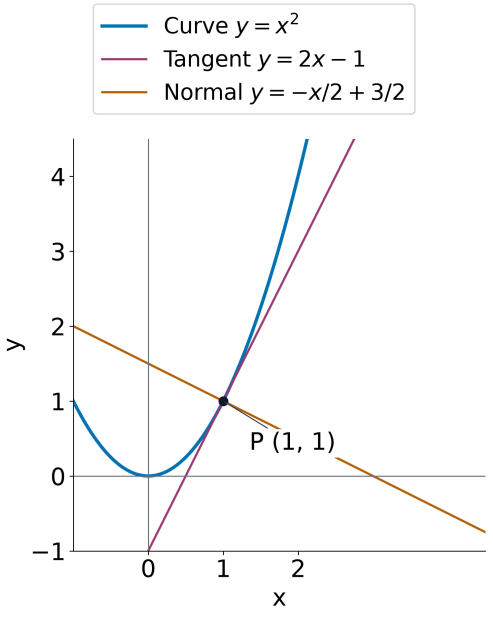
■ Chords, rational powers and a nested root

For $y = x^3$, the gradient of the chord from $x = 2$ to $x = 2 + h$ is $[(2 + h)^3 - 8]/h = 12 + 6h + h^2$ for $h \neq 0$. As $h \to 0$, this tends to 12, the tangent gradient. This is an informal explanation of the derivative; do not substitute zero into the original quotient.

For $x > 0$, $f = 3\sqrt{x} + 2/x$ gives $f' = \frac{3}{2}x^{-1/2} - 2x^{-2}$ and $f'' = -\frac{3}{4}x^{-3/2} + 4x^{-3}$. For a nested function $y = \sqrt{2x^3 + 5}$, use the chain rule to obtain $dy/dx = 3x^2/\sqrt{2x^3 + 5}$ where the root's argument is positive.

Write the tangent and the normal

Find the point as well as the gradient. On $y = x^2$ at $(1, 1)$, $dy/dx = 2$, giving tangent $y - 1 = 2(x - 1)$ and normal $y - 1 = -\frac{1}{2}(x - 1)$. If a point (p, q) has gradient zero, its tangent is $y = q$ and its normal is $x = p$.



The normal is perpendicular to the tangent. Equal axis scales preserve the visible right angle.

From stationary points to a sketch

For $f = x^3 - 6x^2 + 9x + 2$, $f' = 3(x - 1)(x - 3)$, so stationary points have $x = 1, 3$. Substitute into the original function: $(1, 6)$ is a maximum because $f''(1) = -6$; $(3, 2)$ is a minimum because $f''(3) = 6$. The derivative is positive for $x < 1$ and $x > 3$, and negative for $1 < x < 3$. A sketch also needs intercepts and any domain restrictions.

Connected rates and optimisation

If $A = \pi r^2$, then $dA/dt = (dA/dr)(dr/dt) = 2\pi r\, dr/dt$. At $r = 5$ cm and $dr/dt = 0.3$ cm/s, $dA/dt = 3\pi$ cm^2/s. Start with a negative rate for a decreasing quantity. Derivatives with respect to a length and with respect to time are different rates.

For a rectangle beside a wall, 60 m of fence covers two widths x and a length y. The constraint is $2x + y = 60$. Therefore $A = x(60 - 2x)$ with $0 < x < 30$. Solve $A' = 60 - 4x = 0$, giving $x = 15, y = 30$. Since $A'' = -4 < 0$, the maximum area is 450 m^2. Check the physical domain and its boundaries before accepting an optimum.

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Integration

Integration 积分 is the reverse of differentiation. Reversing the power rule gives

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad (n \neq -1,\ a \neq 0).$$

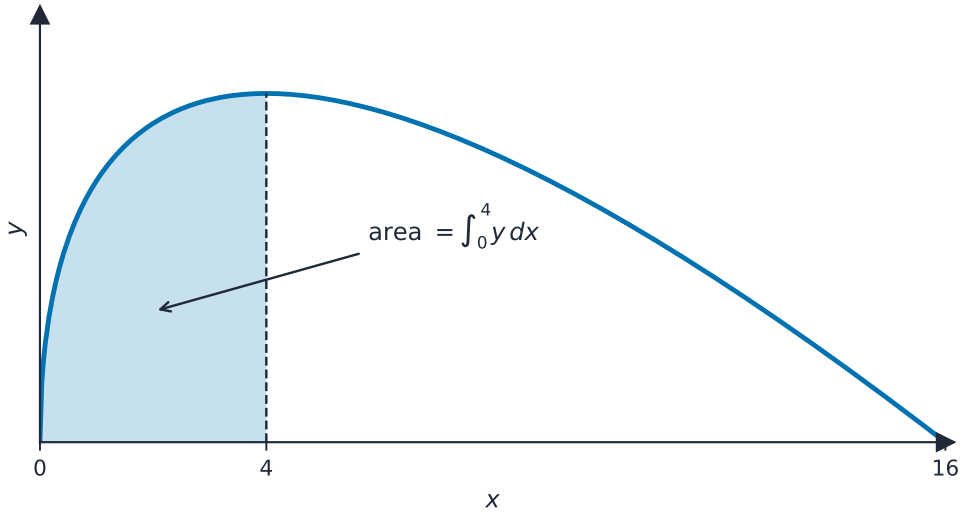
The $+C$ is the **constant of integration** 积分常数. If you know one point on the curve, substitute it to find C.

■ Definite integrals 定积分 and area

A **definite integral** has limits and gives a number:

$$\int_p^q f(x) \, dx = \big[F(x)\big]_p^q = F(q) - F(p).$$

For $p < q$, a definite integral counts contributions above the x-axis positively and those below negatively. It equals geometric area only when the curve is nonnegative throughout. For total area, split at the roots and add the magnitudes of each integral. For the area between two curves, integrate (top curve − bottom curve), splitting wherever their order changes. Reversing the limits reverses the sign of the integral.



$\int_a^b f(x)\,dx =$ net signed area between $y = f(x)$ and the x-axis

The definite integral $\int_0^4 y\,dx$ is the shaded area under the curve.

Worked example. The curve $y = 4x^{1/2} - x$ meets the x-axis again at $x = 16$. Find the area between the curve and the x-axis from $x = 0$ to $x = 4$.

$$\int_0^4 \left(4x^{1/2} - x\right) dx = \left[\frac{8}{3}x^{3/2} - \frac{x^2}{2}\right]_0^4 = \frac{8}{3}(8) - \frac{16}{2} = \frac{64}{3} - 8 = \frac{40}{3}.$$

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An **improper integral** 广义积分 has an infinite limit or an endpoint where the integrand is undefined; evaluate it as a **limit**. For example,

$$\int_1^\infty x^{-2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1,$$

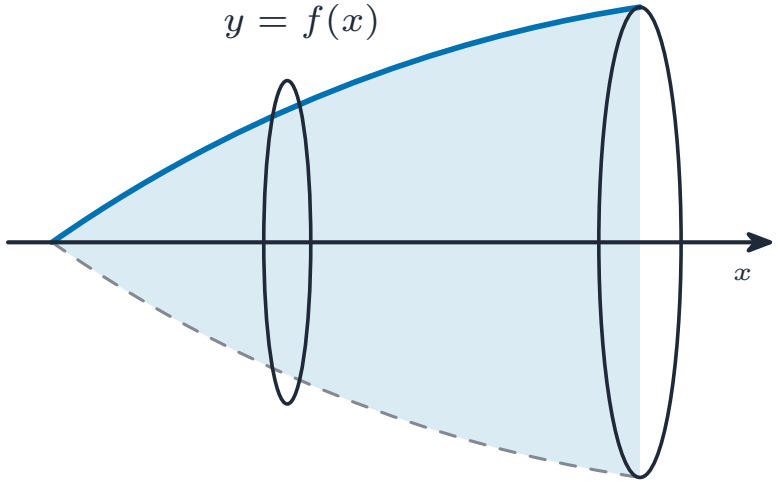
and $\displaystyle\int_0^1 x^{-1/2}\,dx = \lim_{a\to 0^+}\left[2x^{1/2}\right]_a^1 = 2$, which is finite even though the integrand blows up at $x=0$.

■ Volume of revolution 旋转体体积

When a region is turned all the way around an axis it sweeps out a solid. The **volume of revolution** about the x-axis is

$$V = \pi\int_p^q y^2\,dx,$$

and about the y-axis it is $V=\pi\displaystyle\int x^2\,dy$. For example, the region under $y=\sqrt{x}$ from $x=0$ to $x=4$, turned about the x-axis, has volume $\pi\displaystyle\int_0^4 x\,dx = \pi\left[\tfrac{x^2}{2}\right]_0^4 = 8\pi$.



rotate the region around the x-axis to sweep a solid

Rotating the region under a curve around the x-axis sweeps out a solid; each thin slice is a disc of area πy^2, so $V=\pi\int y^2\,dx$

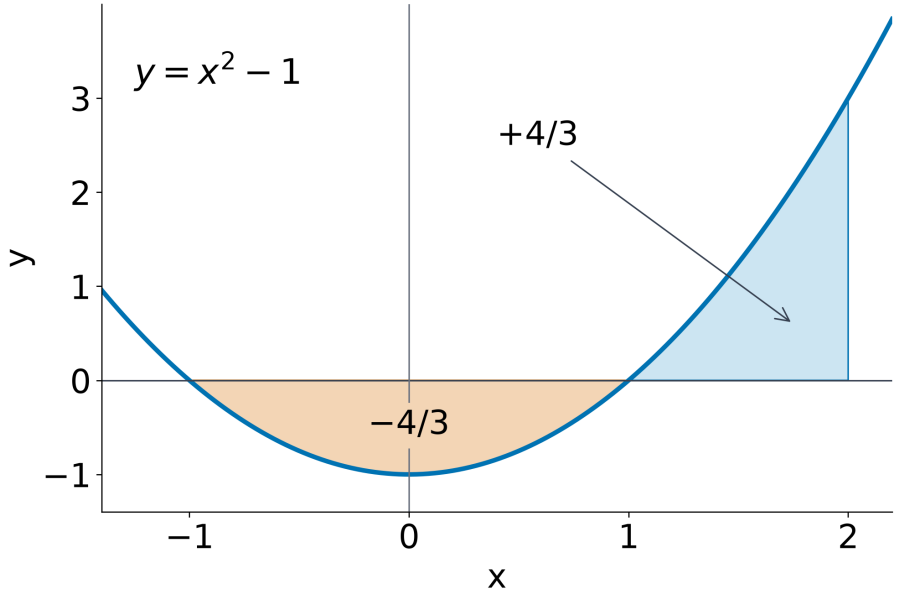
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A curve from two conditions

If $y'' = 6x$ and $(1, 4)$ is a stationary point, integrate once to $y' = 3x^2 + A$. The gradient is zero there, so $A = -3$. Integrate again to $y = x^3 - 3x + B$; using the point gives $B = 6$. Two integrations need two separately determined constants. Check the result by differentiating.

Signed total and geometric area

For $y = x^2 - 1$ from $x = -1$ to 2, the signed integral is zero. Split at $x = 1$: the contribution on $[-1, 1]$ is $-4/3$, and the contribution on $[1, 2]$ is $4/3$. The geometric area is $4/3 + 4/3 = 8/3$ square units.



The integral can vanish even when the total geometric area is positive.

An improper integral is finite only if its limiting value exists as a real number. For example, $\int_0^1 x^{-2}dx = \lim_{a\to 0^+}(1/a - 1)$ has no finite value. The infinity symbol is a bound described by a limit, not a number to substitute.

A gap between curves or a tangent and a curve

The curves $y = 2x$ and $y = x^2$ intersect at $x = 0, 2$. The line is above the parabola between them, so the enclosed area is $\int_0^2 (2x - x^2)dx = 4/3$ square units. If the upper and lower curves exchange places, split the integral at that intersection.

For $y = 2x - x^2$, the tangent at $(1, 1)$ is $y = 1$. On $0 \le x \le 1$, the area under the curve is $2/3$ and the rectangle beneath the tangent has area 1. The gap is $1 - 2/3 = 1/3$. Find a tangent's equation and relevant intercepts before selecting the bounds in a compound area question.

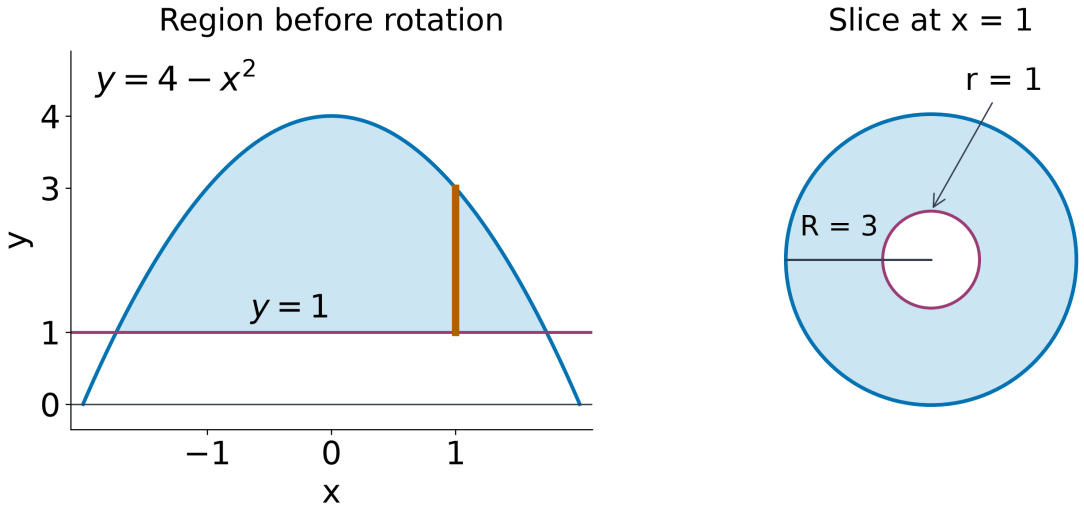
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■ Rotation with a hole and rotation about either axis

A region need not touch the axis of rotation. A slice then has outer radius R and inner radius r; its cross-sectional area is $\pi(R^2 - r^2)$, not $\pi(R - r)^2$.

Between $y = 4 - x^2$ and $y = 1$, intersections occur at $x = \pm\sqrt{3}$. Rotating about the x-axis gives

$$V = \pi \int_{-\sqrt{3}}^{\sqrt{3}} [(4 - x^2)^2 - 1]dx = \frac{88\pi\sqrt{3}}{5}.$$



At $x = 1$ the outer radius is 3 and the inner radius is 1.

About the y-axis, use horizontal slices and express radii in terms of y: $V = \pi \int (R(y)^2 - r(y)^2)dy$. The region $0 \le x \le \sqrt{y}$, $0 \le y \le 4$, produces $V = \pi \int_0^4 y\,dy = 8\pi$. Bounds must be y values when integrating with respect to y. Include any inner radius, and state cubic units.

Exam tips

- Show **every line of algebra** – method marks are lost by jumping steps; use the **discriminant** $b^2 - 4ac$ to decide the number of real roots.
- Work in **radians** for circular measure (arc $= r\theta$, sector area $= \frac{1}{2}r^2\theta$).
- For **differentiation**, set $\frac{dy}{dx} = 0$ for stationary points and use the second derivative to classify them.
- For **integration**, add $+c$ to an indefinite integral and use limits for area; an area below the axis gives a negative integral.

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Pure Mathematics 2

A-Level Mathematics

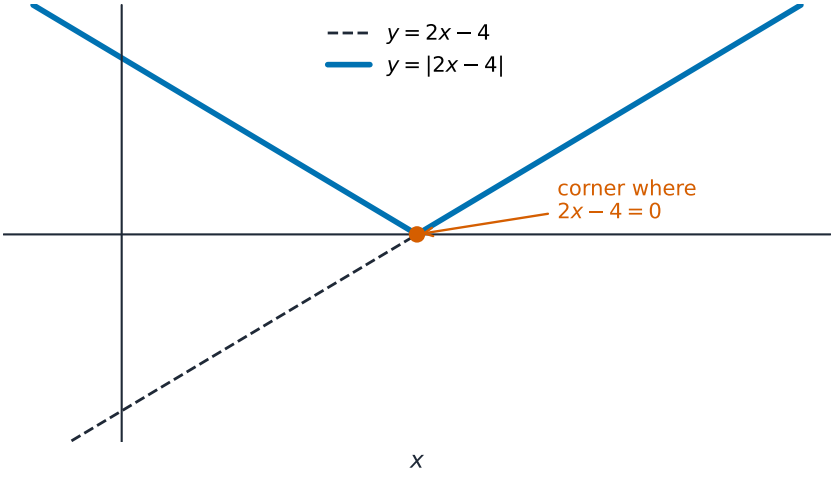
This handout covers Topic 2: **Pure Mathematics** 纯数学 2. It adds the modulus and polynomial algebra, **logarithms** 对数 and the **exponential function** 指数函数, more trigonometry, and new ways to differentiate and integrate.

Algebra

The modulus

The **modulus** 绝对值 $|x|$ is the size of a number with its sign removed, so $|x| \geqslant 0$ always. For $a \neq 0$, the graph of $y = |ax+b|$ is a "V" shape that bounces off the x-axis. Two useful rules for solving equations and inequalities are

$$|a| = |b| \;\Leftrightarrow\; a^2 = b^2, \qquad |x-a| < b \;\Leftrightarrow\; a-b < x < a+b.$$



$y = |f(x)|$ reflects parts of $y = f(x)$ below the x-axis above it

Taking the modulus folds the part of the line below the axis upward into a V.

Worked example. Solve $|3x + 8| < 9$.

Using the second rule with the inequality written as $-9 < 3x + 8 < 9$:

$$-9 < 3x + 8 < 9 \;\Rightarrow\; -17 < 3x < 1 \;\Rightarrow\; -\tfrac{17}{3} < x < \tfrac{1}{3}.$$

Worked example: modulus on both sides. Solve $|x - 2| > |2x + 1|$.

Both sides are never negative, so squaring keeps the inequality the right way round:

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$(x-2)^2 > (2x+1)^2 \;\Rightarrow\; x^2-4x+4 > 4x^2+4x+1 \;\Rightarrow\; 0 > 3x^2+8x-3 = (3x-1)(x+3).$

The quadratic is negative between its roots, so $-3 < x < \frac{1}{3}$. Check $x=0$: $|-2| = 2 > 1 = |1|$, true.

Common mistake. Solving only $x-2 > 2x+1$ (giving $x < -3$) misses the other cases. Squaring, or sketching both V graphs, catches them all.

■ Polynomial division and the factor and remainder theorems

A **polynomial** 多项式 is a finite sum of terms with non-negative integer powers of x, such as $2x^4+3x^2-5$. Its **degree** 次数 is the highest power. When you divide one polynomial by another you get a **quotient** 商 and a **remainder** 余数.

- **Remainder theorem** 余数定理: the remainder when $p(x)$ is divided by $(x-a)$ is $p(a)$.
- **Factor theorem** 因式定理: $(x-a)$ is a factor of $p(x)$ exactly when $p(a)=0$.

> **Worked example.** The polynomial $p(x) = 2x^4+kx^3+kx^2+17x+18$ has factor $(x+2)$. Find k.

By the factor theorem $p(-2)=0$:

$$2(16)+k(-8)+k(4)+17(-2)+18 = 0 \;\Rightarrow\; 16-4k=0 \;\Rightarrow\; k=4.$$

> **Worked example: dividing by a quadratic.** Find the quotient and remainder when $2x^3-3x^2+4x-5$ is divided by x^2+1.

Divide the leading terms: $2x^3 \div x^2 = 2x$. Subtract $2x(x^2+1) = 2x^3+2x$ to leave $-3x^2+2x-5$. Next, $-3x^2 \div x^2 = -3$; subtract $-3(x^2+1) = -3x^2-3$ to leave $2x-2$. That has lower degree than x^2+1, so stop:

$$2x^3-3x^2+4x-5 = (x^2+1)(2x-3)+(2x-2).$$

The quotient is $2x-3$ and the remainder is $2x-2$. Dividing by a quadratic can leave a linear remainder, not just a number.

Logarithmic and exponential functions

A **logarithm** answers the question "what power?". If $a^x = y$ then $x = \log_a y$. For real logarithms, the base is positive and not 1, and the argument is positive. Logarithms and **indices** 指数 (powers) are reverse ideas. The **laws of logarithms** 对数定律 are

$$\log(mn) = \log m + \log n, \qquad \log\frac{m}{n} = \log m - \log n, \qquad \log(m^k) = k\log m.$$

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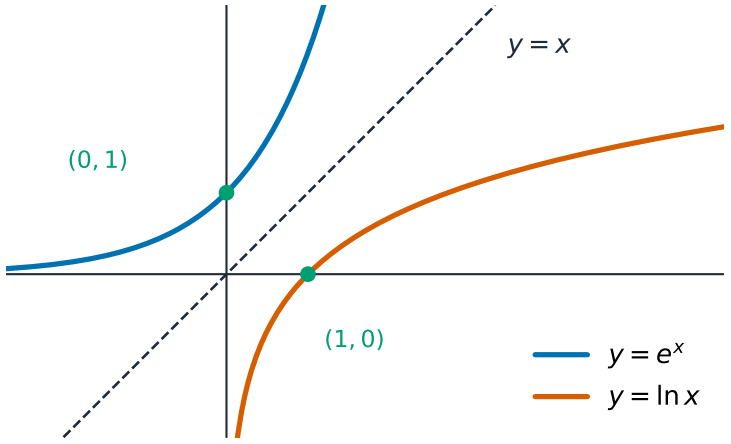
Earthquake strength is measured on the logarithmic Richter scale.



Populations can grow exponentially when resources are plentiful.

The exponential function e^x and the **natural logarithm** 自然对数 $\ln x$ (for $x > 0$) are inverse functions, so $\ln(e^x) = x$ and $e^{\ln x} = x$. When the unknown is in the power, take logs of both sides.

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ln and exp inverse · reflect in $y = x$

e^x and $\ln x$ undo each other, so each is the other reflected in $y = x$.

Worked example. Solve $4^x < 0.05$.

$$4^x < 0.05 \;\Rightarrow\; x \ln 4 < \ln 0.05 \;\Rightarrow\; x < \frac{\ln 0.05}{\ln 4} = -2.16 \text{ (3 s.f.)}.$$

Linear form 线性形式: a relationship like $y = Ax^n$ becomes a straight line if you take logs: $\ln y = \ln A + n \ln x$. Plotting $\ln y$ against $\ln x$ gives a line with gradient n and intercept $\ln A$, so you can find the unknown constants.

Worked example: unknowns in two powers. Solve $2^{x+1} = 3^x$.

Take logs of both sides and bring the powers down: $(x + 1) \ln 2 = x \ln 3$. Collect the x terms: $\ln 2 = x(\ln 3 - \ln 2)$, so

$$x = \frac{\ln 2}{\ln 3 - \ln 2} = 1.71 \text{ (3 s.f.)}.$$

Worked example: a hidden quadratic. Solve $e^{2x} - 5e^x + 6 = 0$, giving exact answers.

Write $e^{2x} = (e^x)^2$ and let $u = e^x$: $u^2 - 5u + 6 = 0$, so $(u - 2)(u - 3) = 0$ and $e^x = 2$ or $e^x = 3$. Hence $x = \ln 2$ or $x = \ln 3$. An equation such as $e^x = -1$ would have **no** solution, because $e^x > 0$ for every x.

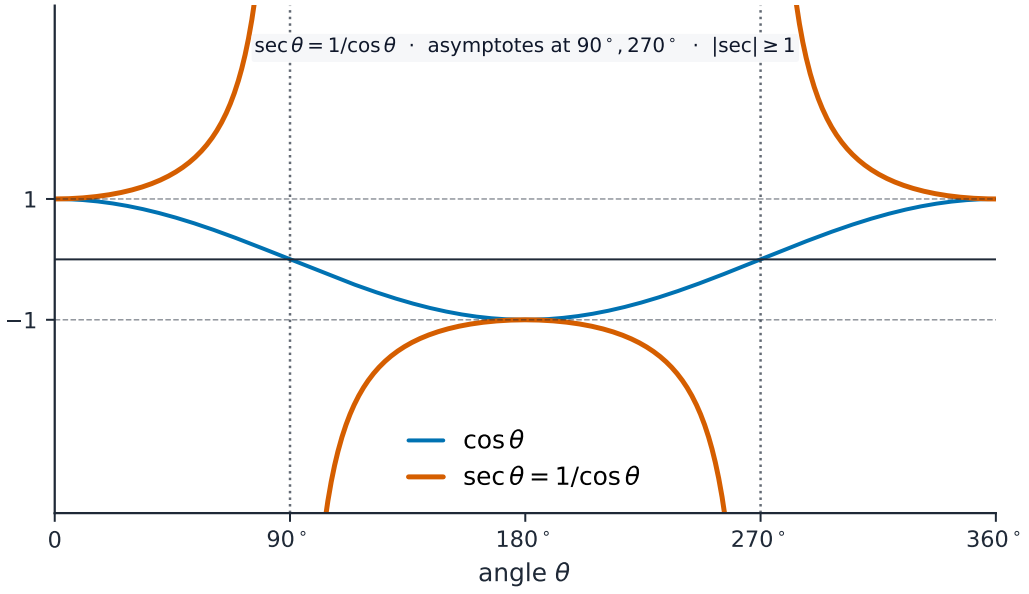
Worked example: finding constants from a line. Two variables satisfy $y = Ax^n$. The graph of $\ln y$ against $\ln x$ is a straight line through $(0.5, 2.1)$ and $(2.5, 5.1)$. Find A and n.

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The gradient is $n = \dfrac{5.1 - 2.1}{2.5 - 0.5} = 1.5$. Then $\ln A = 2.1 - 1.5(0.5) = 1.35$, so $A = e^{1.35} = 3.86$. Remember to undo the log: giving $A = 1.35$ is a common slip.

Trigonometry

There are three more functions, each the reciprocal of one you know: the **secant** 正割 $\sec\theta = \dfrac{1}{\cos\theta}$, the **cosecant** 余割 $\csc\theta = \dfrac{1}{\sin\theta}$, and the **cotangent** 余切 $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ where $\sin\theta \neq 0$. It equals $1/\tan\theta$ where both expressions are defined.



$\sec\theta = 1/\cos\theta$ *rises to an asymptote wherever* $\cos\theta = 0$.

You must know these **trigonometric identities** 三角恒等式 and choose the right one for each problem:

$$\sec^2\theta \equiv 1 + \tan^2\theta, \qquad \csc^2\theta \equiv 1 + \cot^2\theta.$$

You also use the **compound angle** 复合角 formulae for $\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$, the **double angle** 二倍角 formulae

$$\sin 2A = 2\sin A\cos A, \quad \cos 2A = 2\cos^2 A - 1, \quad \tan 2A = \frac{2\tan A}{1 - \tan^2 A},$$

and the **R-formula** 辅助角公式 $a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)$, where $R = \sqrt{a^2 + b^2}$ and $R\cos\alpha = a$, $R\sin\alpha = b$. Choose the quadrant from both signs; $\tan\alpha = b/a$ alone is insufficient, and if $a = b = 0$ the expression is zero.

> **Worked example.** Solve $2\tan^2\theta + 3\sec\theta = 18$ for $-180° < \theta < 180°$.

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Replace $\tan^2\theta$ with $\sec^2\theta - 1$ to get one function:

$$2(\sec^2\theta - 1) + 3\sec\theta = 18 \;\Rightarrow\; 2\sec^2\theta + 3\sec\theta - 20 = 0 \;\Rightarrow\; (2\sec\theta - 5)(\sec\theta + 4) = 0.$$

So $\sec\theta = \frac{5}{2}$ or $\sec\theta = -4$, giving $\cos\theta = \frac{2}{5}$ or $\cos\theta = -\frac{1}{4}$. The solutions are $\theta = \pm 66.4°$ and $\theta = \pm 104.5°$.

Worked example: the R-formula. Express $3\sin\theta + 4\cos\theta$ as $R\sin(\theta + \alpha)$, and solve $3\sin\theta + 4\cos\theta = 2$ for $0° < \theta < 360°$.

Expand: $R\sin(\theta + \alpha) = R\cos\alpha\sin\theta + R\sin\alpha\cos\theta$. Compare: $R\cos\alpha = 3$ and $R\sin\alpha = 4$. So $R = \sqrt{3^2 + 4^2} = 5$ and $\tan\alpha = \frac{4}{3}$, giving $\alpha = 53.13°$.

Now $5\sin(\theta + 53.13°) = 2$, so $\sin(\theta + 53.13°) = 0.4$. The angle $\theta + 53.13°$ lies between $53.13°$ and $413.13°$, so its values are $180° - 23.58° = 156.42°$ and $360° + 23.58° = 383.58°$ (the basic value $23.58°$ is too small). Subtract $53.13°$:

$$\theta = 103.3° \quad \text{or} \quad \theta = 330.4°.$$

Shifting the interval before listing the angles is the step most often missed.

Worked example: an exact value. Find the exact value of $\cos 75°$.

Write $75° = 45° + 30°$ and use $\cos(A + B) = \cos A\cos B - \sin A\sin B$:

$$\cos 75° = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Worked example: proving an identity. Prove that $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$.

Start from the more complicated side and use the double-angle formulae, choosing the form of $\cos 2\theta$ that cancels the 1:

$$\frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Work on one side only, and end with exactly the other side; do not treat the identity as an equation to be solved.

Differentiation

Learn these standard derivatives:

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\ln x = \frac{1}{x}, \quad \frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}\tan x = \sec^2 x.$$

For a product or a quotient of two functions, use:

- the **product rule** 乘积法则: $(uv)' = u'v + uv'$;

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- the **quotient rule** 商法则: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

When a curve is given by **parametric equations** 参数方程 $x = x(t)$, $y = y(t)$, the gradient is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. When y is defined **implicitly** 隐式 (not made the subject), differentiate every term with respect to x, using the chain rule on the y terms, then solve for $\frac{dy}{dx}$.

Worked example. Given $y = 6x\cos(x^2+1)$, find $\frac{dy}{dx}$.

Use the product rule with $u = 6x$ and $v = \cos(x^2+1)$ (and the chain rule for v):

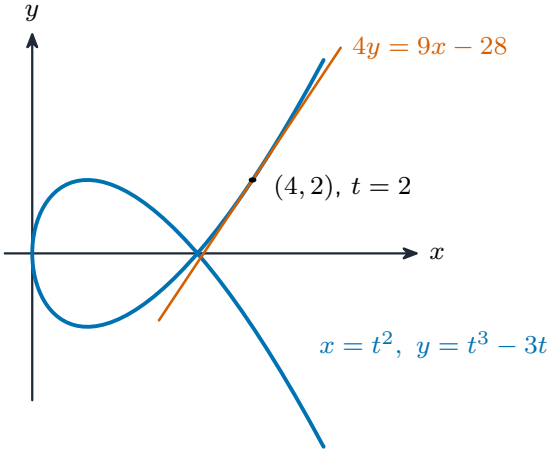
$$\frac{dy}{dx} = 6\cos(x^2+1) + 6x\cdot\big(-2x\sin(x^2+1)\big) = 6\cos(x^2+1) - 12x^2\sin(x^2+1).$$

Worked example: a parametric tangent. A curve has $x = t^2$ and $y = t^3 - 3t$. Find the equation of the tangent at the point where $t = 2$.

Differentiate each with respect to t: $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 3$, so

$$\frac{dy}{dx} = \frac{3t^2-3}{2t} = \frac{9}{4} \quad \text{at } t = 2.$$

At $t = 2$ the point is $(4, 2)$, so the tangent is $y - 2 = \frac{9}{4}(x - 4)$, that is $4y = 9x - 28$. Use the point $(x, y) = (4, 2)$, not $(t, \ldots) = (2, \ldots)$.



The tangent goes through the point (4, 2), not through $t = 2$.

Worked example: a stationary point with the quotient rule. Find the stationary point of $y = \frac{e^{2x}}{x}$ for $x > 0$, and decide its nature.

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With $u = e^{2x}$ and $v = x$:

$$\frac{dy}{dx} = \frac{2e^{2x} \cdot x - e^{2x} \cdot 1}{x^2} = \frac{e^{2x}(2x-1)}{x^2}.$$

e^{2x} is never zero, so $\dfrac{dy}{dx} = 0$ only when $x = \frac{1}{2}$, where $y = \dfrac{e}{1/2} = 2e$. Just below $x = \frac{1}{2}$ the gradient is negative, and just above it is positive, so $(\frac{1}{2}, 2e)$ is a **minimum**.

> **Worked example: an implicit curve.** Find the equation of the **normal** to $x^2 + 2xy - y^2 = 7$ at the point $(2, 1)$.

Differentiate every term with respect to x; the product rule on $2xy$ gives $2y + 2x\dfrac{dy}{dx}$:

$$2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx} = \frac{x+y}{y-x}.$$

At $(2, 1)$ the gradient of the tangent is $\dfrac{3}{-1} = -3$, so the normal has gradient $\frac{1}{3}$: $y - 1 = \frac{1}{3}(x - 2)$, that is $x - 3y + 1 = 0$.

Integration

Integration is **reverse differentiation** – reverse each new derivative; harder integrals may need **integration by substitution** 换元积分 (developed in Pure 3). For a linear inside function $(ax + b)$ with $a \neq 0$:

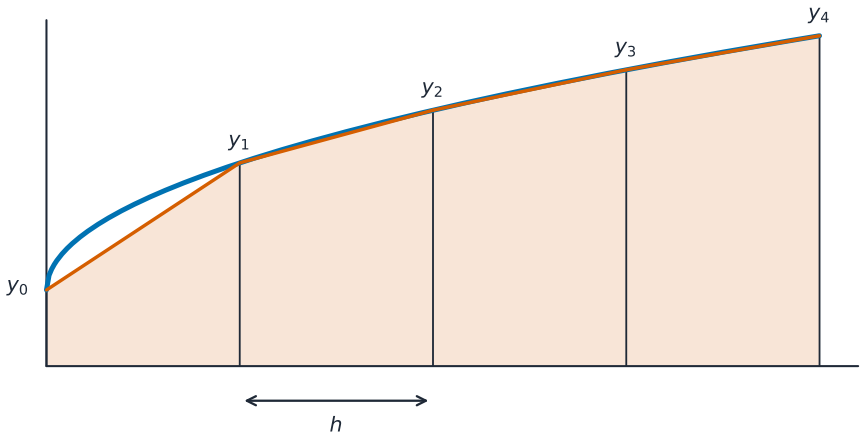
$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C, \qquad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C,$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C, \qquad \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C, \qquad \int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C.$$

To integrate a power of sin or cos, first use an identity to remove the power. When you cannot integrate exactly, the **trapezium rule** 梯形法则 estimates a definite integral:

$$\int_a^b y\,dx \approx \tfrac{h}{2}\big[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\big].$$

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$$\int_a^b y\,dx \approx \tfrac{h}{2}(y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n)$$

Each strip of width h is a trapezium; their areas add up to estimate the integral.

Worked example. Find $\displaystyle\int 6\sin^2 x\,dx$.

Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$:

$$\int 6\sin^2 x\,dx = \int (3 - 3\cos 2x)\,dx = 3x - \tfrac{3}{2}\sin 2x + C.$$

Worked example: an exact definite integral. Find $\displaystyle\int_1^3 \frac{1}{2x+1}\,dx$ and $\displaystyle\int_0^{\ln 2} e^{2x}\,dx$ exactly.

$$\int_1^3 \frac{1}{2x+1}\,dx = \Big[\tfrac{1}{2}\ln(2x+1)\Big]_1^3 = \tfrac{1}{2}(\ln 7 - \ln 3) = \tfrac{1}{2}\ln\tfrac{7}{3},$$

$$\int_0^{\ln 2} e^{2x}\,dx = \Big[\tfrac{1}{2}e^{2x}\Big]_0^{\ln 2} = \tfrac{1}{2}(e^{2\ln 2} - 1) = \tfrac{1}{2}(4-1) = \tfrac{3}{2},$$

using $e^{2\ln 2} = e^{\ln 4} = 4$.

Worked example: the trapezium rule. Use the trapezium rule with 2 intervals to estimate $\displaystyle\int_0^1 \sqrt{1+x^3}\,dx$, and say whether it is an overestimate or an underestimate.

With $h = 0.5$ the ordinates are $y_0 = 1$, $y_1 = \sqrt{1.125} = 1.0607$ and $y_2 = \sqrt{2} = 1.4142$:

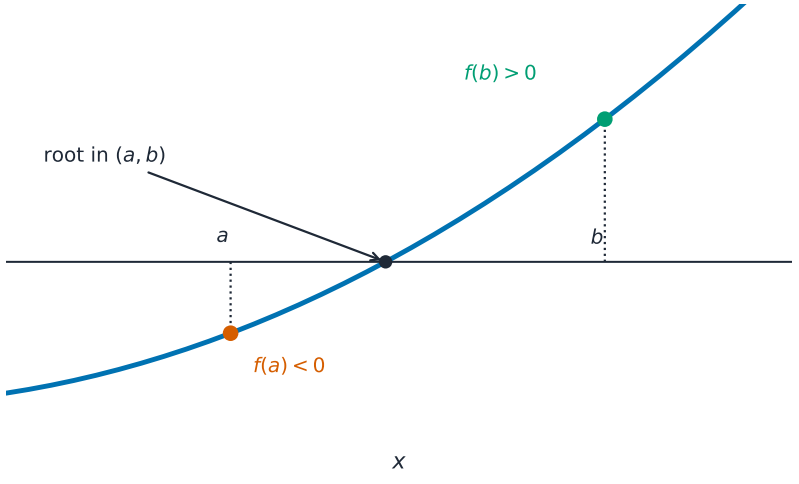
$$\int_0^1 \sqrt{1+x^3}\,dx \approx \frac{0.5}{2}\big[1 + 1.4142 + 2(1.0607)\big] = 1.13.$$

The curve bends upward (it is convex) on $[0, 1]$, so each straight top lies **above** the curve, and 1.13 is an overestimate. (The true value is 1.11.) Remember: the number of ordinates is one more than the number of intervals.

Numerical solution of equations

Many equations cannot be solved exactly. Two ideas help you find a **root** 根 (a solution).

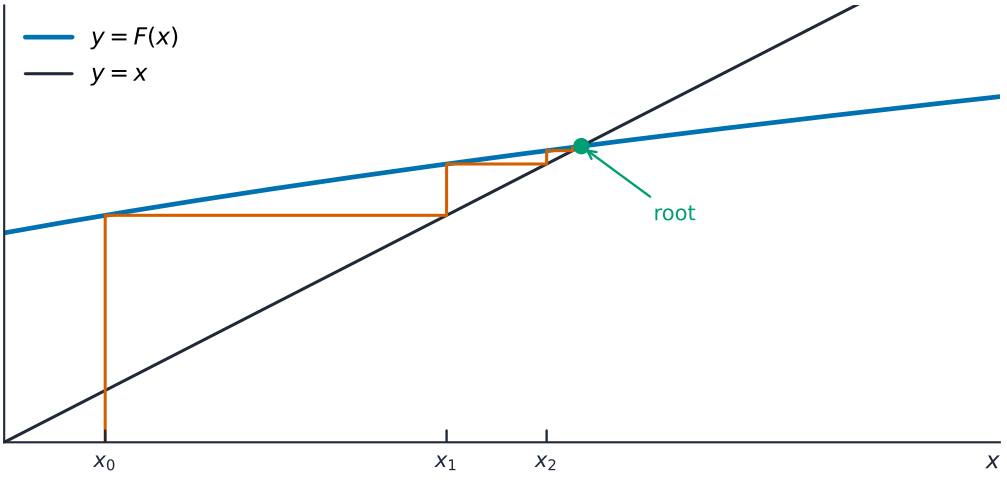
- **Sign change** 变号: if $f(a)$ and $f(b)$ have opposite signs (and the graph has no break between them), a root lies between a and b.
- **Iteration** 迭代: rearrange the equation into the form $x = F(x)$, then use the **iterative formula** 迭代公式 $x_{n+1} = F(x_n)$. Start from a first guess x_0 and repeat. If the values are **convergent** they settle down and **converge** 收敛 to a root. Keep going until the answer is steady to the accuracy asked for.



if f continuous on $[a, b]$ and $f(a)f(b) < 0$, then a root lies in (a, b)

For a continuous function on $[a, b]$, opposite endpoint signs trap a root between a and b.

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$x_{n+1} = F(x_n)$ · converges near the root if $|F'(x)| < 1$ there

Each step goes up to $y = F(x)$ then across to $y = x$; the staircase closes in on the root.

Worked example. A root β of an equation satisfies $x = \sqrt[3]{-2x - 4.5}$, and $-1.4 < \beta < -1.0$. Use the iteration $x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ with $x_0 = -1.2$.

$x_1 = \sqrt[3]{-2(-1.2) - 4.5} = \sqrt[3]{-2.1} = -1.281, \qquad x_2 = \sqrt[3]{-2(-1.281) - 4.5} = -1.247, \quad \ldots$

The values settle near -1.26, so $\beta = -1.26$ (3 s.f.).

How the iteration relates to the equation. At the limit, $x_{n+1} = x_n = \beta$, so $\beta = \sqrt[3]{-2\beta - 4.5}$. Cubing gives $\beta^3 = -2\beta - 4.5$, so β is a root of $x^3 + 2x + 4.5 = 0$. An exam part such as "find the equation satisfied by the limit" wants exactly this step. To check the accuracy of -1.26, show a sign change of $f(x) = x^3 + 2x + 4.5$ between -1.265 and -1.255: $f(-1.265) = -0.054 < 0$ and $f(-1.255) = +0.013 > 0$, so the root lies between them and is -1.26 to 3 s.f.

Exam tips

- Use the **laws of logarithms** to solve equations; remember ln and e^x are inverses.
- For **numerical methods**, show a **sign change** to locate a root, set out the iteration clearly, and give the answer to the stated accuracy.
- Learn the **chain, product and quotient rules** and identify which the function needs.
- Solve modulus equations $|f(x)| = g(x)$ by considering both the positive and negative cases, and sketch to check.
- For "exact" answers keep surds, ln and e: $\cos 75° = \frac{1}{4}(\sqrt{6} - \sqrt{2})$, not 0.259.

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- In an identity proof, work on **one** side until it becomes the other.

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Pure Mathematics 3

A-Level Mathematics

This handout covers Topic 3: **Pure Mathematics** 纯数学 3. Subtopics 3.1–3.6 build on the algebra, logarithms, trigonometry, differentiation, integration and numerical methods of Pure Mathematics 2, so this handout explains what is **new** in Pure 3 and then covers the three big new areas: vectors, differential equations and complex numbers.

Algebra

Two new tools join the algebra from Pure 2.

■ Partial fractions

A single fraction with a factorised bottom can be split into a sum of simpler fractions. This is called writing it in **partial fractions** 部分分式, and it makes a **rational function** 有理函数 (a fraction of polynomials) easy to integrate or expand. Match the form to the bottom:

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}, \qquad \frac{1}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}.$$

First check the top is **lower degree** than the bottom. If it is not ("top-heavy"), **divide** the polynomial first: dividing gives a **quotient** 商 plus a **remainder** 余数 over the original bottom, and you split only that remaining proper fraction.

$$\frac{x+4}{(x+1)(x-2)} \xrightarrow{\text{split}} \frac{2}{x-2} - \frac{1}{x+1}$$

One fraction with a factorised bottom splits into a sum of simpler fractions.

Worked example. Express $\dfrac{x+4}{(x+1)(x-2)}$ in partial fractions.

Write $\dfrac{x+4}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$, so $x+4 = A(x-2) + B(x+1)$. Put $x=2$: $6 = 3B$, so $B=2$. Put $x=-1$: $3 = -3A$, so $A=-1$. Hence

$$\frac{x+4}{(x+1)(x-2)} = \frac{2}{x-2} - \frac{1}{x+1}.$$

Worked example: a repeated factor. Express $\dfrac{5x+4}{(x-1)(x+2)^2}$ in partial fractions.

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A squared factor needs **two** fractions, one over $(x+2)$ and one over $(x+2)^2$:

$$\frac{5x+4}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2},$$

so $5x+4 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$. Put $x=1$: $9 = 9A$, so $A=1$. Put $x=-2$: $-6 = -3C$, so $C=2$. No value of x removes both A and C, so compare the x^2 terms instead: $0 = A+B$, so $B=-1$. Hence

$$\frac{5x+4}{(x-1)(x+2)^2} = \frac{1}{x-1} - \frac{1}{x+2} + \frac{2}{(x+2)^2}.$$

Check with $x=0$: the left side is $\dfrac{4}{(-1)(4)} = -1$ and the right side is $-1-\frac{1}{2}+\frac{1}{2} = -1$.

Common mistakes. Leaving out the $\dfrac{B}{x+2}$ term for a squared factor, and splitting a top-heavy fraction without dividing first.

■ Polynomials and the modulus (from Pure 2)

The **factor theorem** 因式定理 says that $(x-a)$ is a factor of $p(x)$ exactly when $p(a)=0$. The **remainder theorem** 余数定理 says that dividing $p(x)$ by $(x-a)$ leaves the remainder $p(a)$.

> **Worked example.** $p(x) = 2x^3 + ax^2 - 13x + 6$ has a factor $(x-2)$. Find a, factorise $p(x)$ fully, and find the remainder when $p(x)$ is divided by $(x+1)$.

$p(2) = 16 + 4a - 26 + 6 = 4a - 4 = 0$, so $a=1$. Dividing by $(x-2)$ gives $p(x) = (x-2)(2x^2+5x-3) = (x-2)(2x-1)(x+3)$. The remainder on dividing by $(x+1)$ is $p(-1) = -2+1+13+6 = 18$.

> **Worked example: a modulus inequality.** Solve $|2x-1| < |x+1|$.

Both sides are never negative, so squaring keeps the inequality the same way round:

$$4x^2 - 4x + 1 < x^2 + 2x + 1 \;\Rightarrow\; 3x^2 - 6x < 0 \;\Rightarrow\; 3x(x-2) < 0.$$

So $0 < x < 2$. A sketch of $y = |2x-1|$ and $y = |x+1|$ shows the same thing: the V-shaped graphs cross at $x=0$ and $x=2$.

■ The binomial expansion for a rational power

The **binomial expansion** 二项展开式 also works when the power is a fraction or is negative, as long as $|x| < 1$:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

For example $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$ for $|x| < 1$.

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Worked example. Expand $(4-x)^{-1/2}$ as far as the term in x^2, and state the values of x for which it is valid.

The formula needs a 1 in front, so take out the 4 first, raised to the same power:

$$(4-x)^{-1/2} = 4^{-1/2}\left(1-\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}.$$

Now use $n=-\frac{1}{2}$ with $-\frac{x}{4}$ in place of x:

$$\left(1-\frac{x}{4}\right)^{-1/2} = 1+\left(-\tfrac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\cdots = 1+\frac{x}{8}+\frac{3x^2}{128}+\cdots$$

Multiply by $\frac{1}{2}$: $(4-x)^{-1/2} = \dfrac{1}{2}+\dfrac{x}{16}+\dfrac{3x^2}{256}+\cdots$. It is valid when $\left|\frac{x}{4}\right|<1$, that is $|x|<4$.

Common mistakes. Writing $4(1-\frac{x}{4})^{-1/2}$ (the 4 must also be raised to the power $-\frac{1}{2}$); losing the minus sign inside $\left(-\frac{x}{4}\right)^2$ in later terms; and giving the validity as $|x|<1$ instead of $|x|<4$.

Logarithms, trigonometry and numerical methods (from Pure 2)

Subtopics 3.2, 3.3 and 3.6 are the same skills you met in Pure 2: the laws of logarithms with e^x and $\ln x$; the identities $\sec^2\theta \equiv 1+\tan^2\theta$ and $\csc^2\theta \equiv 1+\cot^2\theta$, the compound- and double-angle formulae, and the R-form of $a\sin\theta+b\cos\theta$; and solving an equation numerically by a **sign change** 变号 and an **iterative formula** 迭代公式 $x_{n+1} = F(x_n)$. Use them exactly as before.

The three new trig functions are the **reciprocals** 倒数 of the familiar ones: the **secant** 正割 $\sec\theta = \dfrac{1}{\cos\theta}$, the **cosecant** 余割 $\csc\theta = \dfrac{1}{\sin\theta}$, and the **cotangent** 余切 $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ for $\sin\theta \neq 0$. It equals $1/\tan\theta$ wherever both expressions are defined. (Memory aid: match the *third* letter – se**c** goes with **c**osine.) The two Pythagorean identities above come straight from dividing $\sin^2\theta+\cos^2\theta \equiv 1$ by $\cos^2\theta$ or $\sin^2\theta$. For a numerical method, an iteration $x_{n+1}=F(x_n)$ **converges** 收敛 if its values approach a finite limit. Smaller successive changes alone do not prove convergence; when F is continuous at that limit, the limit is a fixed point. Check it in the original equation.

Worked example: reducing to linear form. Two variables are thought to obey $y=kx^n$. A graph of $\ln y$ against $\ln x$ is a straight line through $(0.5, 2.1)$ and $(2.0, 4.5)$. Find k and n.

Take logs: $\ln y = n\ln x + \ln k$. This is $Y = nX + c$ with $Y=\ln y$ and $X=\ln x$, so n is the **gradient** 斜率 and $\ln k$ is the **intercept** 截距:

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$$n = \frac{4.5 - 2.1}{2.0 - 0.5} = 1.6, \qquad \ln k = 2.1 - 1.6 \times 0.5 = 1.3, \qquad k = e^{1.3} = 3.67.$$

For $y = Ab^x$ instead, plot $\ln y$ against x (not $\ln x$): $\ln y = x \ln b + \ln A$, so the gradient is $\ln b$ and the intercept $\ln A$.

> **Worked example: the R-form.** Express $3\sin\theta + 4\cos\theta$ as $R\sin(\theta + \alpha)$, and solve $3\sin\theta + 4\cos\theta = 2$ for $0° < \theta < 360°$.

Expand: $R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$, so $R\cos\alpha = 3$ and $R\sin\alpha = 4$. Then $R = \sqrt{3^2 + 4^2} = 5$ and $\tan\alpha = \frac{4}{3}$, $\alpha = 53.13°$.
Now $\sin(\theta + 53.13°) = 0.4$. The range for $\theta + 53.13°$ is $53.13°$ to $413.13°$, so $\theta + 53.13° = 156.42°$ or $383.58°$, giving $\theta = 103.3°$ or $330.4°$. The greatest value of the expression is $R = 5$.

> **Worked example: locating a root and iterating.** Show that $x^3 + x - 3 = 0$ has a root between 1 and 2, and use $x_{n+1} = \sqrt[3]{3 - x_n}$ with $x_1 = 1.2$ to find it to 3 decimal places.

Let $f(x) = x^3 + x - 3$. $f(1) = -1 < 0$ and $f(2) = 7 > 0$: f is continuous and changes sign, so there is a root between 1 and 2. The iterates are 1.2164, 1.2127, 1.2136, 1.2134, 1.2134, so the root is 1.213 to 3 d.p. The iterative formula is a rearrangement of the equation: $x = \sqrt[3]{3 - x}$ cubes to $x^3 = 3 - x$.

Differentiation

The methods are those of Pure 2 (the product, quotient and chain rules, with parametric and implicit curves). One derivative is added – the **inverse tangent** 反正切:

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}.$$

> **Worked example.** Differentiate $y = \tan^{-1}(3x)$.

Use the chain rule with the result above: $\dfrac{dy}{dx} = \dfrac{1}{1 + (3x)^2} \times 3 = \dfrac{3}{1 + 9x^2}$.

> **Worked example: a parametric curve.** A curve has $x = t^2 + 1$, $y = t^3 - 3t$. Find the tangent at $t = 2$ and the points where the tangent is horizontal.

For a curve defined **parametrically** 参数地, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 3}{2t}$. At $t = 2$ the point is $(5, 2)$ and the gradient is $\dfrac{9}{4}$, so the tangent is $y - 2 = \frac{9}{4}(x - 5)$. The tangent is horizontal where $3t^2 - 3 = 0$, so $t = \pm 1$: the points $(2, -2)$ and $(2, 2)$.

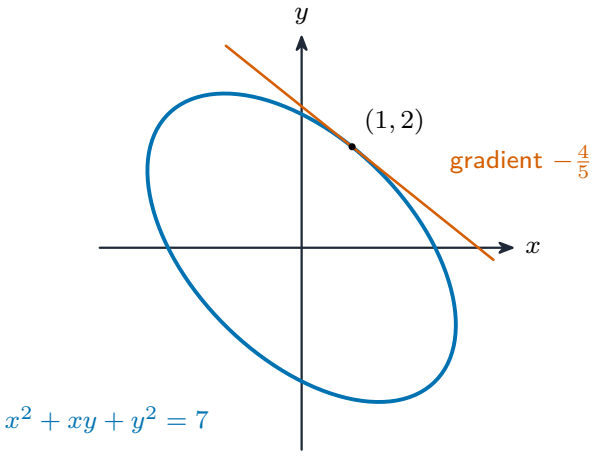
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Worked example: an implicit curve. Find the gradient of $x^2 + xy + y^2 = 7$ at $(1, 2)$.

Differentiate **implicitly** 隐式, using the product rule on xy and the chain rule on y^2:

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

At $(1, 2)$: $\dfrac{dy}{dx} = -\dfrac{4}{5}$. (Check first that the point is on the curve: $1 + 2 + 4 = 7$.)



Implicit differentiation gives the slope of the tangent without solving for y.

Integration

Pure 3 adds several powerful methods.

- A new standard integral: $\displaystyle\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$.
- **Partial fractions**: split a rational function first, then integrate each piece as a logarithm.
- The pattern $\dfrac{k\,f'(x)}{f(x)}$: this integrates to $k\ln|f(x)| + C$. For example $\displaystyle\int \frac{2x}{x^2 + 1}\,dx = \ln(x^2 + 1) + C$.
- **Integration by parts** 分部积分, used for a product: $\displaystyle\int u\,\frac{dv}{dx}\,dx = uv - \int v\,\frac{du}{dx}\,dx$.
- **Integration by substitution** 换元积分: a given change of variable turns a hard integral into an easy one.

Worked example. Find $\displaystyle\int x\cos x\,dx$.

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Use integration by parts with $u = x$ and $\dfrac{dv}{dx} = \cos x$, so $\dfrac{du}{dx} = 1$ and $v = \sin x$:

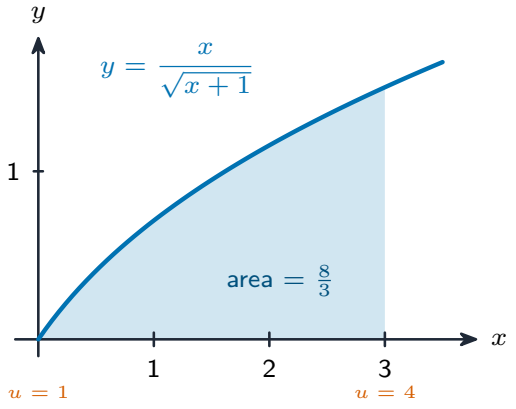
$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Worked example: substitution. Use the substitution $u = x + 1$ to find $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$.

Change all three parts: $x = u - 1$, $dx = du$, and the limits $x = 0$ and $x = 3$ become $u = 1$ and $u = 4$:

$$\int_1^4 \frac{u-1}{\sqrt{u}}\,du = \int_1^4 \left(u^{1/2} - u^{-1/2}\right)du = \left[\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^4 = \left(\tfrac{16}{3} - 4\right) - \left(\tfrac{2}{3} - 2\right) = \frac{8}{3}.$$

The common error is to keep the old limits 0 and 3 after changing to u.



Same area, new variable: the limits change with it.

Worked example: partial fractions. Show that $\displaystyle\int_3^4 \frac{x+4}{(x+1)(x-2)}\,dx = \ln\frac{16}{5}$.

Use the partial fractions found earlier, then integrate each piece as a logarithm:

$$\int_3^4 \left(\frac{2}{x-2} - \frac{1}{x+1}\right)dx = \Big[2\ln(x-2) - \ln(x+1)\Big]_3^4 = (2\ln 2 - \ln 5) - (0 - \ln 4).$$

This is $2\ln 2 + 2\ln 2 - \ln 5 = \ln 16 - \ln 5 = \ln\dfrac{16}{5}$. Because the answer is given, show the log laws step by step.

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Vectors



Forces like wind and water are vectors – they have both size and direction.

A **vector** 向量 has both size and direction. Write it as a column, or as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, or as $\overrightarrow{AB}$.

- The **magnitude** 模长 (length) of $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$.
- A **unit vector** 单位向量 has magnitude 1; divide a nonzero vector by its magnitude to make one.
- A **position vector** 位置向量 gives a point's place from the origin; a **displacement vector** 位移向量 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ goes from one point to another. Multiplying by a **scalar** 标量 (a plain number) stretches a vector.

Adding vectors means following one, then the other ("nose to tail"). If $\overrightarrow{AB} = k\overrightarrow{CD}$, the lines AB and CD are **parallel**, and AB is k times as long. The midpoint of AB has position vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

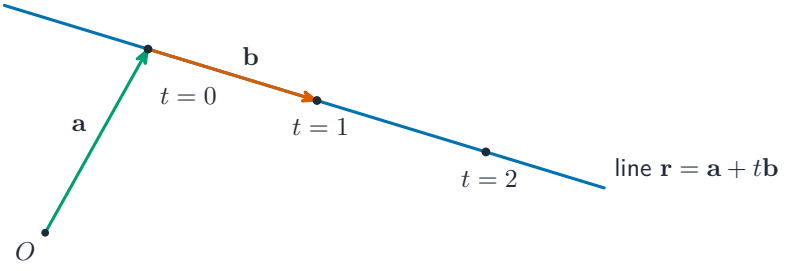
> **Worked example: geometry with vectors.** $OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. Show that its diagonals bisect each other.

$\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$ (along OA, then AB, which equals OC). The midpoint M of the other diagonal AC has $\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}(\mathbf{c} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}\overrightarrow{OB}$. So M is also the midpoint of OB: the diagonals cut each other in half.

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■ Lines and the scalar product

A straight line through point **a** in direction **b** has vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. Two lines may be **parallel** 平行, may **intersect** 相交 at a point, or may be **skew lines** 异面直线 (not parallel and never meeting).

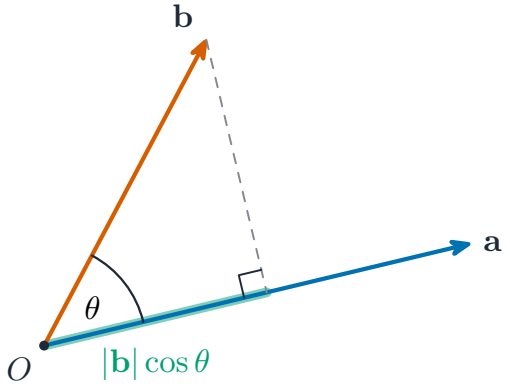


Start at the point **a**, *then add* t *copies of the direction* **b** *to reach any point on the line.*

The **scalar product** 数量积 (dot product) of **a** and **b** is

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ is the angle between them. So $\mathbf{a} \cdot \mathbf{b} = 0$ means the vectors are perpendicular.



The scalar product picks out $|\mathbf{b}| \cos\theta$, *how far* **b** *reaches along* **a**.

Worked example. Find the angle between $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

$$\mathbf{a} \cdot \mathbf{b} = (1)(2) + (2)(2) + (2)(1) = 8, \qquad |\mathbf{a}| = |\mathbf{b}| = 3.$$

So $\cos\theta = \dfrac{8}{3 \times 3} = \dfrac{8}{9}$, giving $\theta = 27.3°$.

Worked example: do two lines meet? Line l is $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and line m is $\mathbf{r} = -\mathbf{j} + \mathbf{k} + s(3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$. Show that they intersect and find the point.

The directions are not multiples of each other, so the lines are not parallel. Use a different letter for each line's parameter and set the components equal:

$$x:\ 1+t=3s, \qquad y:\ 3-t=-1+2s, \qquad z:\ 2t=1+3s.$$

Add the x and y equations: $4=-1+5s$, so $s=1$ and then $t=2$. Now test the third equation: $2(2)=4$ and $1+3(1)=4$. It holds, so the lines meet, at $\mathbf{i}+3\mathbf{j}+2(\mathbf{i}-\mathbf{j}+2\mathbf{k})=3\mathbf{i}+\mathbf{j}+4\mathbf{k}$, the point $(3,1,4)$.

If the third equation had failed, the lines would be skew. Always use the equation you did **not** solve as the check.

> **Worked example: the foot of the perpendicular.** Find the point F on the line $\mathbf{r}=\mathbf{i}+\mathbf{k}+t(\mathbf{i}+\mathbf{j}+\mathbf{k})$ nearest to $P(4,1,3)$, and the shortest distance from P to the line.

A general point of the line is $(1+t,\ t,\ 1+t)$, so $\overrightarrow{PF}=(t-3)\mathbf{i}+(t-1)\mathbf{j}+(t-2)\mathbf{k}$. At the nearest point, $\overrightarrow{PF}$ is **perpendicular** 垂直 to the direction $\mathbf{i}+\mathbf{j}+\mathbf{k}$:

$$(t-3)+(t-1)+(t-2)=0 \ \Rightarrow\ t=2.$$

So $F=(3,2,3)$, $\overrightarrow{PF}=-\mathbf{i}+\mathbf{j}$, and the shortest distance is $|\overrightarrow{PF}|=\sqrt{2}$.

Differential equations

A **differential equation** 微分方程 links a quantity to its **rate of change**. To solve a first-order equation whose variables are **separable** 可分离变量, put all the y terms on one side and all the x terms on the other, then integrate both sides. This gives the **general solution** 通解, which contains a constant. An **initial condition** 初始条件 (a known value) fixes the constant and gives the **particular solution** 特解.

> **Worked example.** Solve $\dfrac{dy}{dx}=xy$, given that $y=1$ when $x=0$.

Separate the variables and integrate:

$$\int \frac{1}{y}\,dy=\int x\,dx \ \Rightarrow\ \ln y=\tfrac{1}{2}x^2+c \ \Rightarrow\ y=Ae^{x^2/2}.$$

Using $y=1$ at $x=0$ gives $A=1$, so $y=e^{x^2/2}$.

> **Worked example: forming the equation.** Water leaks from a tank. The depth h cm decreases at a rate proportional to $\sqrt{h}$. At $t=0$ minutes the depth is 16 cm, and at $t=10$ it is 9 cm. When is the tank empty?

"Decreases at a rate proportional to $\sqrt{h}$" gives $\dfrac{dh}{dt}=-k\sqrt{h}$, with $k>0$. Separate and integrate:

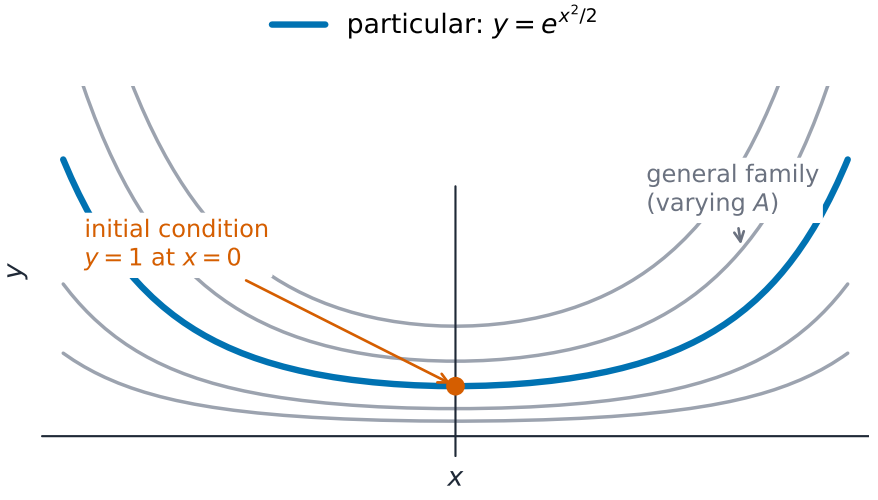
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$$\int h^{-1/2}\,dh = \int -k\,dt \;\Rightarrow\; 2\sqrt{h} = -kt + c.$$

At $t = 0$: $2\sqrt{16} = c$, so $c = 8$. At $t = 10$: $2\sqrt{9} = -10k + 8$, so $k = 0.2$. The tank is empty when $h = 0$: $0 = -0.2t + 8$, so $t = 40$ minutes.

Interpreting the solution. The solution is $\sqrt{h} = 4 - 0.1t$, so $h = (4 - 0.1t)^2$ for $0 \le t \le 40$. The depth falls quickly at first and more slowly as the tank empties, because the rate depends on $\sqrt{h}$. The model stops at $t = 40$: for larger t the formula would make h rise again, which is impossible. An exam part such as "what happens to h as t becomes large?" wants a sentence like this, in the context of the problem.

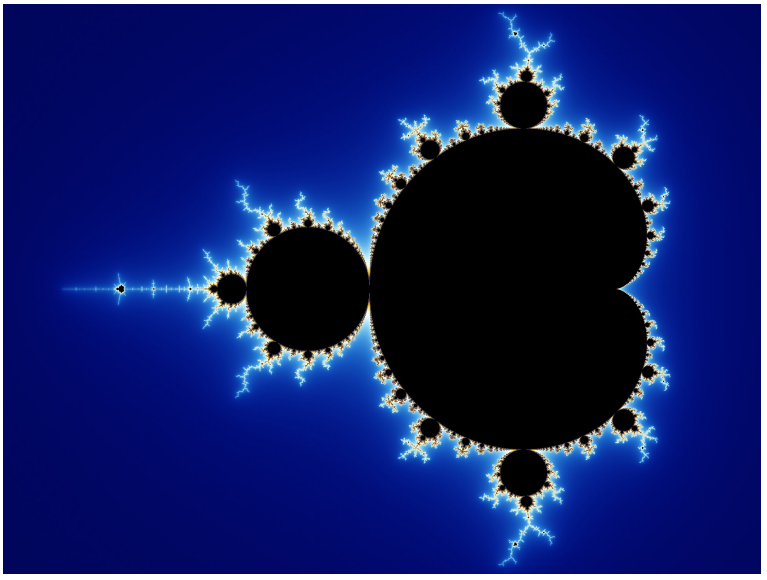
Common mistakes. Leaving out the minus sign for a decrease, and adding the constant only after rearranging (so writing $y = e^{x^2/2} + c$ instead of $y = Ae^{x^2/2}$ in the example above).



The constant A gives a whole family of curves; the condition $y = 1$ at $x = 0$ selects $y = e^{x^2/2}$.

Complex numbers

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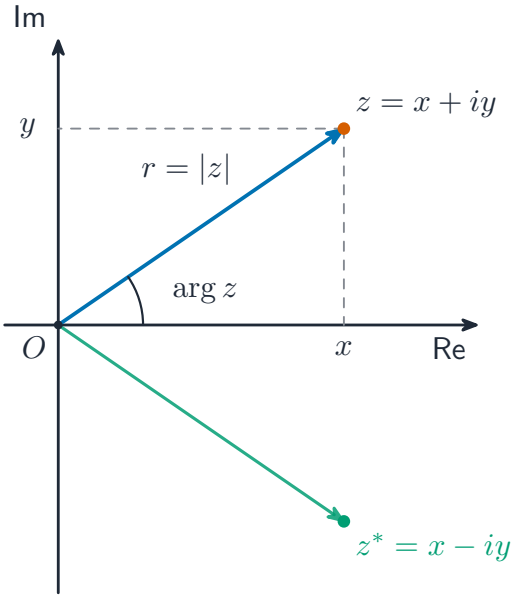


The Mandelbrot set: the complex numbers c for which repeating $z \to z^2 + c$, starting from $z = 0$, stays bounded

A **complex number** 复数 has the form $z = x + iy$, where $i^2 = -1$. Here x is the **real part** 实部 and y is the **imaginary part** 虚部. This $x + iy$ is the **Cartesian form** 直角坐标形式. Two complex numbers are equal only when their real parts match and their imaginary parts match.

- The **conjugate** 共轭 of $z = x + iy$ is $z^* = x - iy$. For a polynomial with real coefficients, any non-real roots come in conjugate pairs.
- The **modulus** 模 is $|z| = \sqrt{x^2 + y^2}$ (its distance from the origin) and, for $z \neq 0$, the **argument** 辐角 is the angle the point makes, measured from the positive real axis. Zero has no argument.
- You can plot z as a point on an **Argand diagram** 阿干图 (the complex plane).
- The **polar form** 极坐标形式 is $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where $r = |z|$ and θ is the argument. Multiplying multiplies the moduli and adds the arguments.
- The **loci** of points satisfying a condition on z are drawn on the Argand diagram – e.g. $|z - a| = r$ is a circle, $|z - a| = |z - b|$ a perpendicular bisector, and $\arg(z - a) = \theta$ a half-line. The **square roots** of a complex number come from solving $w^2 = z$.

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On the Argand diagram $|z|$ is the distance from O, $\arg z$ the angle, and z^* the reflection in the real axis.

To divide, multiply top and bottom by the conjugate of the bottom.

Worked example: polar form. $z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $z_2 = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. Find $z_1 z_2$ and $\dfrac{z_1}{z_2}$.

Multiply the moduli and **add** the arguments: $z_1 z_2 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 8i$. Divide the moduli and **subtract** the arguments: $\dfrac{z_1}{z_2} = \frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \dfrac{\sqrt{3}}{4} + \dfrac{1}{4}i$.

Worked example. Write $\dfrac{3+i}{1-i}$ in the form $x + iy$.

$$\frac{3+i}{1-i} = \frac{(3+i)(1+i)}{(1-i)(1+i)} = \frac{3+3i+i+i^2}{1+1} = \frac{2+4i}{2} = 1+2i.$$

Worked example: modulus and argument. Find the modulus and argument of $z = -1 + \sqrt{3}\,i$.

$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$. The point $(-1, \sqrt{3})$ is in the **second** quadrant, so draw it before using a calculator. The angle with the negative real axis is $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$, so $\arg z = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Hence $z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$.

A calculator's $\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ gives $-\frac{\pi}{3}$, which is the wrong quadrant. In the same way $\arg(-1-i) = -\frac{3\pi}{4}$, not $\frac{\pi}{4}$.

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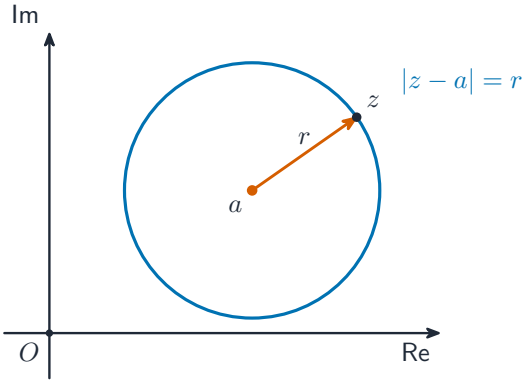
Worked example: a square root. Find the square roots of $3+4i$.

Let $(a+bi)^2 = 3+4i$ with a and b real. Then $a^2 - b^2 + 2abi = 3+4i$, so compare real and imaginary parts:

$$a^2 - b^2 = 3, \qquad 2ab = 4.$$

Substitute $b = \dfrac{2}{a}$: $a^2 - \dfrac{4}{a^2} = 3$, so $a^4 - 3a^2 - 4 = 0$, giving $(a^2-4)(a^2+1) = 0$. Since a is real, $a^2 = 4$, so $a = 2$ with $b = 1$, or $a = -2$ with $b = -1$. The square roots are $\pm(2+i)$. Check: $(2+i)^2 = 4 + 4i - 1 = 3 + 4i$.

An equation or inequality in z describes a **locus** 轨迹 (a path or region) on the Argand diagram. For example $|z-a| = r$ is a circle of radius r centred at a.



$|z-a| = r$ is the set of points a fixed distance r from a – a circle.

Exam tips

- Split a rational function into **partial fractions** before integrating or expanding.
- Choose the right **integration technique** (substitution, by parts, or partial fractions) from the form of the integrand.
- Use the scalar (dot) product for the **angle between vectors** and to test for perpendicularity; write a line as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.
- Give **complex numbers** in the form asked for (Cartesian or modulus-argument) and show them on an Argand diagram.
- When an answer is "exact", leave ln, e, π and surds in it; give decimals only when asked.
- To find k and n from a log graph, write the log equation in the form $Y = mX + c$ first, then read the gradient and intercept.
- For the foot of a perpendicular, write a general point of the line with its parameter, then set the scalar product with the direction to zero.

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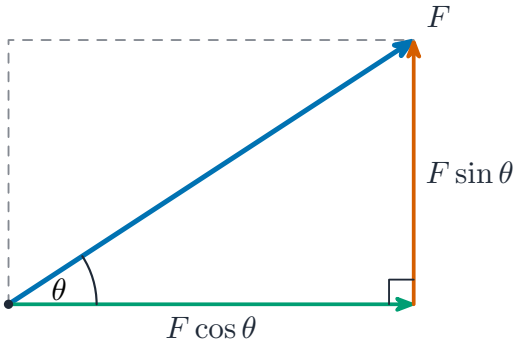
Mechanics

A-Level Mathematics

This handout covers Topic 4: **Mechanics** 力学. It studies how forces make objects move. Throughout this topic, take the acceleration of free fall as $g = 10$ m s^{-2}.

Forces and equilibrium

A **force** 力 is a push or a pull. It is a vector, so it has size and direction. Because it is a vector, you can split a force into **components** 分量 (usually horizontal and vertical), and you can add several forces into one **resultant** 合力.



A force at angle θ has a horizontal part $F\cos\theta$ and a vertical part $F\sin\theta$.

A particle is in **equilibrium** 平衡 when the forces are balanced: the vector sum of the forces is zero. In practice this means the components in any direction add to zero.

Drawing a force diagram. Draw the body as a dot or a simple box and add an arrow for every force acting **on** it, each labelled:

- its **weight** mg, straight down (mass is in kg; weight is a force, in newtons);
- a **normal reaction** R at right angles to each surface it touches;
- **friction** F along a rough surface, opposite to the way the body moves or would move;
- a **tension** T along each string, pulling away from the body;
- any applied force, driving force or resistance given in the question.

Forces the body exerts on other things do **not** go on its diagram. Then choose two perpendicular directions to resolve in: horizontal and vertical, or along and at right angles to a slope.

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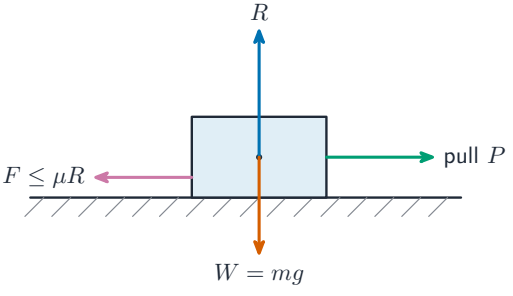
Every mechanics problem is a picture like this. Three forces act on the climber – weight straight down, tension along the rope, and a push from the rock – and because they balance, the climber hangs still in equilibrium. Resolving each into horizontal and vertical parts turns the picture into equations

Friction

When two surfaces touch, the **contact force** 接触力 between them has two parts: the **normal reaction** 法向反作用力 R, at right angles to the surface, and the **friction** 摩擦力 F, along the surface, which opposes sliding. A "smooth" surface is a model with no friction.

Friction can only grow up to a maximum. At that maximum the body is in **limiting equilibrium** 极限平衡, about to slip, and the friction is **limiting friction** 最大静摩擦力. The maximum is set by the **coefficient of friction** 摩擦系数 μ:

$$F \leqslant \mu R, \qquad \text{with } F = \mu R \text{ at the point of slipping.}$$



On a rough surface the contact force splits into the normal reaction R and friction F (at most μR).

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By **Newton's third law** 牛顿第三定律, the two surfaces push on each other with equal and opposite forces.

> **Worked example.** A block of weight 20 N rests on a rough horizontal table with coefficient of friction $\mu = 0.4$. Find the largest horizontal force that can be applied before the block slides.

The table's normal reaction balances the weight, so $R = 20$ N. The block is on the point of slipping when friction reaches its maximum $F = \mu R = 0.4 \times 20 = 8$ N. In equilibrium the applied force equals the friction, so the largest force it can resist is 8 N.

> **Worked example: resolving.** A particle of weight 20 N hangs in equilibrium from two strings. One string makes 30° with the horizontal and the other makes 60°, on opposite sides. Find the two tensions T_1 and T_2.

Resolve horizontally and vertically; each direction must sum to zero:

$$\rightarrow:\ T_2\cos 60^\circ = T_1\cos 30^\circ, \qquad \uparrow:\ T_1\sin 30^\circ + T_2\sin 60^\circ = 20.$$

The first equation gives $T_2 = T_1\dfrac{\cos 30^\circ}{\cos 60^\circ} = \sqrt{3}\,T_1$. Substitute: $\frac{1}{2}T_1 + \sqrt{3}T_1 \cdot \frac{\sqrt{3}}{2} = 2T_1 = 20$, so $T_1 = 10$ N and $T_2 = 10\sqrt{3} = 17.3$ N. The steeper string carries more of the weight, as you would expect.

> **Worked example: will it slip?** A block of mass 4 kg is placed on a rough plane inclined at 30° to the horizontal, with $\mu = 0.3$. Show that it slides, find its acceleration, and find the least force parallel to the slope that would hold it at rest.

Resolve at right angles to the slope: $R = 40\cos 30^\circ = 34.64$ N, so the greatest friction available is $\mu R = 0.3 \times 34.64 = 10.39$ N. The weight component down the slope is $40\sin 30^\circ = 20$ N. This is more than 10.39 N, so friction cannot hold the block: it slides. Down the slope, $F = ma$ gives $20 - 10.39 = 4a$, so $a = 2.40$ m s^{-2}.

To hold it at rest with the **least** force P up the slope, friction helps P and is at its limit, acting up the slope: $P + 10.39 = 20$, so $P = 9.61$ N. (To push it **up** the slope instead, friction would act down the slope, and P would have to exceed $20 + 10.39 = 30.4$ N.)

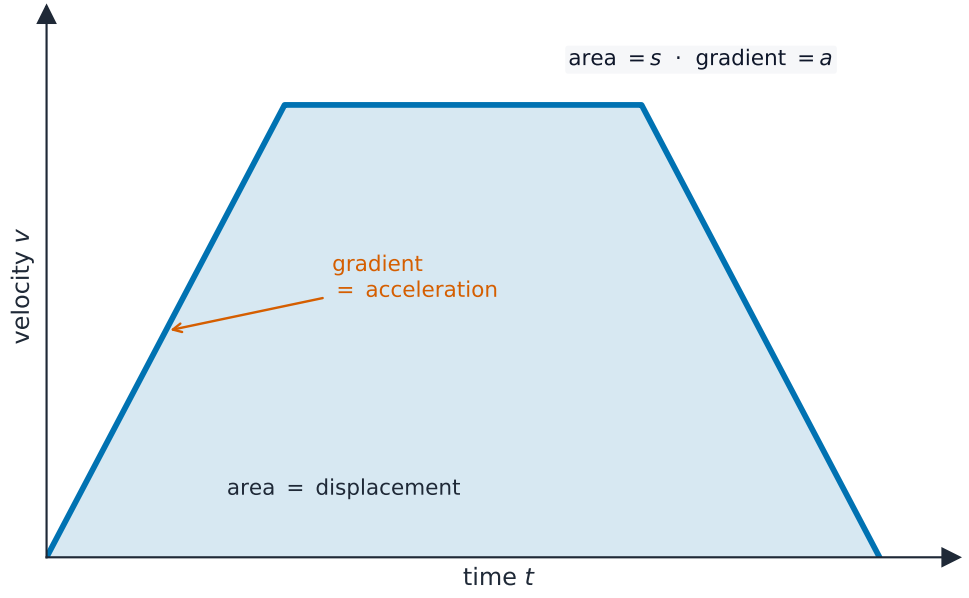
Common mistakes. Mixing up sin and cos: the component **next to** the angle uses cos. Check with the angle you actually drew, not the one in the question. Writing $F = \mu R$ when the body is not on the point of slipping: in ordinary equilibrium the friction is only as large as it needs to be, $F < \mu R$.

Kinematics of motion in a straight line

Distance 距离 and **speed** 速率 are scalars (size only). **Displacement** 位移, **velocity** 速度 and **acceleration** 加速度 are vectors (size and direction).

On a **velocity-time graph** 速度时间图, the area under the graph is the displacement and the gradient is the acceleration (**deceleration** 减速度 means acceleration opposite

to the velocity, reducing speed). On a displacement-time graph, the gradient is the velocity. More generally, differentiate with respect to time to go from displacement to velocity to acceleration, and integrate to go back.



The shaded area gives the distance travelled; the slope of the line gives the acceleration.

For motion with **constant acceleration** 匀加速, use these formulae (the "suvat" equations):

$$v = u + at, \qquad s = ut + \tfrac{1}{2}at^2, \qquad v^2 = u^2 + 2as, \qquad s = \tfrac{1}{2}(u + v)t.$$

Worked example. A car starts from rest and accelerates at 2.5 m s^{-2} for 4 s. Find its speed and the distance travelled.

$$v = 0 + 2.5 \times 4 = 10 \text{ m s}^{-1}, \qquad s = 0 + \tfrac{1}{2}(2.5)(4^2) = 20 \text{ m}.$$

Worked example: a velocity-time graph. A train starts from rest and accelerates uniformly to 20 m s^{-1} in 40 s. It travels at this speed for T seconds, then decelerates uniformly to rest in 20 s. The total distance is 2.4 km. Find T and the deceleration.

Sketch the graph: a trapezium. The distance is the area under it:

$$\tfrac{1}{2}(40)(20) + 20T + \tfrac{1}{2}(20)(20) = 2400 \;\Rightarrow\; 400 + 20T + 200 = 2400 \;\Rightarrow\; T = 90 \text{ s}.$$

The deceleration is the size of the gradient of the last part: $20 \div 20 = 1$ m s^{-2}.

Worked example: vertical motion. A ball is thrown vertically upwards at 15 m s^{-1} from a point 2 m above the ground. Find its greatest height above the ground and the time until it hits the ground.

Take **upwards** as positive, so $a = -10$ m s^{-2}. At the top $v = 0$: $0 = 15^2 - 20s$, so $s = 11.25$ m, and the greatest height is $11.25 + 2 = 13.25$ m. It hits the ground when its displacement is $s = -2$:

$$-2 = 15t - 5t^2 \;\Rightarrow\; 5t^2 - 15t - 2 = 0 \;\Rightarrow\; t = \frac{15 + \sqrt{265}}{10} = 3.13 \text{ s},$$

taking the positive root. One equation covers the whole flight, up and down, as long as the signs are kept.

Worked example: acceleration that changes. A particle moves in a straight line with velocity $v = 6t - t^2$ m s^{-1} for $0 \leqslant t \leqslant 6$. Find its greatest speed and the distance it travels.

The acceleration is not constant, so suvat does **not** apply; use calculus. Differentiate: $a = \dfrac{dv}{dt} = 6 - 2t$, which is zero at $t = 3$, so the greatest speed is $v = 18 - 9 = 9$ m s^{-1}. The velocity is never negative on this interval, so the distance is

$$s = \int_0^6 (6t - t^2)\,dt = \Big[3t^2 - \tfrac{1}{3}t^3\Big]_0^6 = 108 - 72 = 36 \text{ m}.$$

If v changed sign, you would integrate each part separately and add the sizes.

Worked example: when the velocity changes sign. A particle has $v = t^2 - 4t + 3 = (t-1)(t-3)$ for $0 \leqslant t \leqslant 3$. Find its displacement and the total distance travelled.

The particle stops and turns round at $t = 1$. Integrate each part:

$$\int_0^1 (t^2 - 4t + 3)\,dt = \Big[\tfrac{1}{3}t^3 - 2t^2 + 3t\Big]_0^1 = \tfrac{4}{3}, \qquad \int_1^3 (t^2 - 4t + 3)\,dt = 0 - \tfrac{4}{3} = -\tfrac{4}{3}.$$

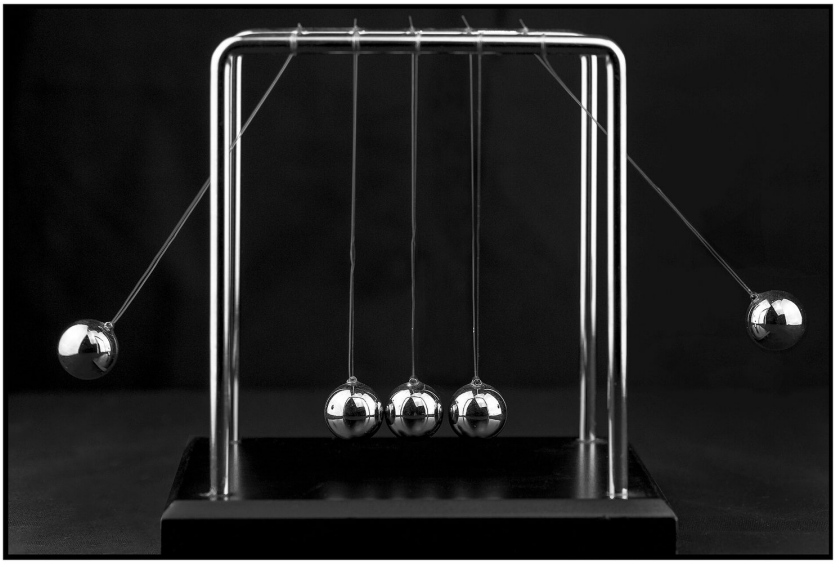
The displacement is $\frac{4}{3} - \frac{4}{3} = 0$: the particle ends where it started. The distance is $\frac{4}{3} + \frac{4}{3} = \frac{8}{3}$ m. A single integral from 0 to 3 would give the displacement, not the distance.

Momentum

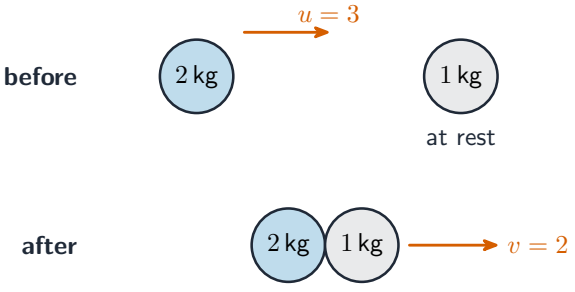
The **linear momentum** 动量 of a body is mass × velocity. It is a vector. In a direct collision of two bodies, the total momentum is unchanged when the net external impulse is zero or negligible. This is the **conservation of linear momentum** 动量

守恒:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2.$$



A Newton's cradle demonstrates conservation of momentum in collisions.



$$2(3) + 1(0) = (2 + 1)\,v$$

The total momentum is the same before and after the collision.

Worked example. A body of mass 2 kg moving at 3 m s^{-1} hits a stationary body of mass 1 kg, and they stick together. Find their common speed afterwards.

$$2(3) + 1(0) = (2 + 1)v \;\Rightarrow\; v = \frac{6}{3} = 2 \text{ m s}^{-1}.$$

Worked example: moving towards each other. A 3 kg body moving at 4 m s^{-1} meets a 2 kg body moving at 5 m s^{-1} in the opposite direction. After they collide, the 3 kg body moves at 0.5 m s^{-1} in its original direction. Find the velocity of the 2 kg body.

Take the 3 kg body's direction as positive, so the other body's velocity before is -5:

$$3(4) + 2(-5) = 3(0.5) + 2v \;\Rightarrow\; 2 = 1.5 + 2v \;\Rightarrow\; v = 0.25 \text{ m s}^{-1}.$$

It is positive, so the 2 kg body now moves in the 3 kg body's original direction. Forgetting the minus sign for the opposite direction is the usual error.

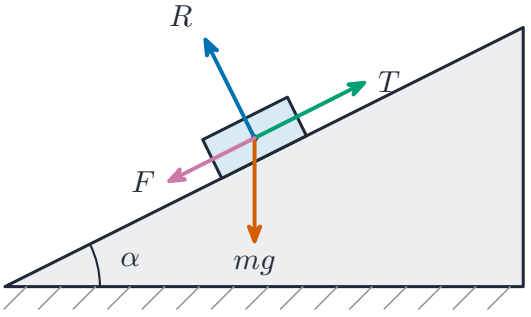
Newton's laws of motion

Newton's laws of motion 牛顿运动定律 connect force and acceleration. The key one is: resultant force = **mass** 质量 × acceleration,

$$F = ma.$$

The **weight** 重力 of a body is the force of gravity on it: $W = mg$. Forces in a problem may include weight, friction, **tension** 张力 in a string, the **thrust** 推力 (push) in a rod, and **air resistance** 空气阻力.

For motion on an **inclined plane** 斜面, split each force into a part along the slope and a part at right angles to it, then apply $F = ma$ along the slope. For **connected particles** 连接质点 (joined by a string), apply $F = ma$ to each body, or to the whole system.



On a slope, resolve the forces along the slope and at right angles to it.

Worked example. A block of mass 12 kg is pulled up a rough plane by a rope parallel to the slope. The plane is at 20° to the horizontal, the coefficient of friction is 0.4, and the acceleration is 2 m s^{-2}. Find the tension in the rope.

The normal reaction is $R = mg\cos 20° = 120\cos 20° = 112.8$ N, so the friction is $F = \mu R = 0.4 \times 112.8 = 45.1$ N. Along the slope, $T - mg\sin 20° - F = ma$:

$$T = ma + mg\sin 20° + F = 12(2) + 120\sin 20° + 45.1 = 24 + 41.0 + 45.1 = 110 \text{ N (3 s.f.)}.$$

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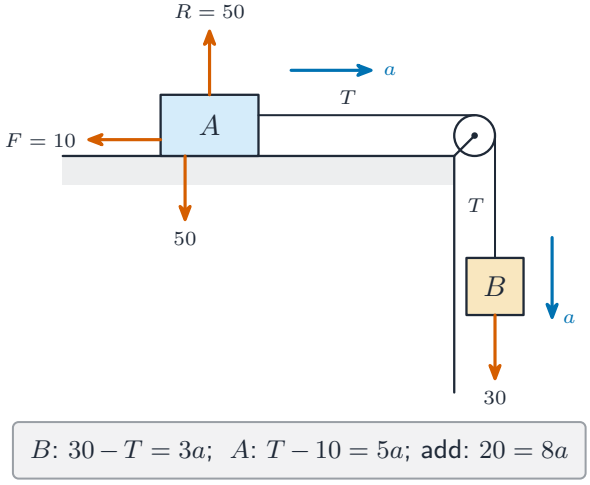
Worked example: a lift. A woman of mass 60 kg stands in a lift that accelerates **upwards** at 0.5 m s^{-2}. Find the normal reaction from the floor. Upwards, $R - 600 = 60(0.5)$, so $R = 630$ N. By Newton's third law she pushes down on the floor with 630 N, more than her weight; if the lift accelerated downwards at 0.5 m s^{-2}, R would be 570 N.

Worked example: connected particles. Block A of mass 5 kg lies on a rough horizontal table ($\mu = 0.2$). A light inextensible string joins it, over a smooth pulley at the table's edge, to block B of mass 3 kg, which hangs freely. The system is released from rest. Find the acceleration and the tension.

For A, $R = 5g = 50$ N, so friction is $0.2 \times 50 = 10$ N. Write $F = ma$ for each block in its own direction of motion:

$$B: \ 30 - T = 3a, \qquad A: \ T - 10 = 5a.$$

Add them: $20 = 8a$, so $a = 2.5$ m s^{-2}, and then $T = 30 - 3(2.5) = 22.5$ N. Check with A: $22.5 - 10 = 12.5 = 5(2.5)$. Note that T is **less** than B's weight; if it were equal, B would not accelerate.



Forces in newtons. Write $F = ma$ for each block in its own direction of motion; adding the two equations removes T.

Energy, work and power

The **work done** 功 by a constant force is the **scalar product** of force and displacement – the force times the distance moved in the direction of the force: $W = Fd\cos\theta$, where θ is the angle between the force and the motion. Work is measured in joules (J).



A roller coaster trades potential energy for kinetic energy as it rises and falls.

Energy comes in forms you can calculate:

- **Kinetic energy** 动能 (energy of movement): $\text{KE} = \frac{1}{2}mv^2$.
- **Gravitational potential energy** 重力势能 (energy of height): $\text{PE} = mgh$.

The net work done on a particle equals its change in kinetic energy. When only gravity does work, kinetic plus gravitational potential energy stays constant. More generally, total energy is conserved when all transfers, including heating by friction and work by other forces, are accounted for – the **conservation of energy** 能量守恒.

Power 功率 is the rate of doing work. For a force pulling in the direction of motion, $P = Fv$ (power = force × velocity). Power is measured in watts (W).

> **Worked example.** A car engine works at 12 kW while the car moves at 20 m s^{-1} on a level road. Find the driving force.
>
> $$P = Fv \;\Rightarrow\; F = \frac{P}{v} = \frac{12000}{20} = 600 \text{ N}.$$

> **Worked example: energy with friction.** A 5 kg box slides from rest 10 m down a slope at 30° to the horizontal. A constant friction force of 12 N acts. Find its speed at the bottom.

The box drops a height $h = 10\sin 30° = 5$ m, so it loses $mgh = 5(10)(5) = 250$ J of potential energy. The work done against friction is $12 \times 10 = 120$ J. The rest becomes kinetic energy:

$$\tfrac{1}{2}(5)v^2 = 250 - 120 = 130 \;\Rightarrow\; v^2 = 52 \;\Rightarrow\; v = 7.21 \text{ m s}^{-1}.$$

Use the **vertical** height for potential energy (5 m) but the distance **along the slope**

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for the work against friction (10 m).

> **Worked example: power on a hill.** A car of mass 1200 kg moves up a hill inclined at α to the horizontal, where $\sin\alpha = 0.05$. The engine works at 30 kW and the resistance to motion is a constant 600 N. Find the **instantaneous** acceleration when the speed is 15 m s^{-1}, and the greatest possible speed on this hill.

At 15 m s^{-1} the driving force is $F = \dfrac{P}{v} = \dfrac{30000}{15} = 2000$ N. Up the slope:

$$2000 - 600 - 1200(10)(0.05) = 1200a \;\Rightarrow\; 800 = 1200a \;\Rightarrow\; a = 0.667 \text{ m s}^{-2}.$$

As the car speeds up, $F = P/v$ falls, so the acceleration falls too. The greatest speed is reached when $a = 0$: then $F = 600 + 600 = 1200$ N, so $v = \dfrac{30000}{1200} = 25$ m s^{-1}.

Exam tips

- Draw a clear **force diagram** and resolve into perpendicular components; for equilibrium, each direction sums to zero.
- Use **SUVAT** only for constant acceleration and keep a consistent positive direction.
- Apply $F = ma$ along the direction of motion, including friction: $F \leqslant \mu R$ for static contact, with equality at limiting friction; use the stated sliding-friction model during sliding.
- State your **assumptions** (light inextensible string, smooth pulley, particle) – they are often worth a mark.
- To find a **distance** from v as a function of t, find where $v = 0$ first and integrate each part separately.
- With a constant power, the driving force is P/v, so it changes with speed; the **greatest** speed is when the driving force just balances the resistances.
- In a vertical-motion question, choose one positive direction and keep $a = -10$ (or $+10$) for the whole flight.

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Probability & Statistics 1

A-Level Mathematics

This handout covers Topic 5: **Probability & Statistics** 概率统计 1. It is about describing data, counting choices, and working out the chance of events.

Representation of data

Choose a diagram that suits the data. You should be able to draw and read:

- a **stem-and-leaf diagram** 茎叶图 (keeps the original values and shows the shape); two data sets are compared with a **back-to-back** 背靠背 version – a shared central stem, one set's leaves increasing to the left and the other's to the right, so you can compare their medians and spreads at a glance;
- a **box-and-whisker plot** 箱线图 (shows the lowest value, the three quartiles, and the highest value);
- a **histogram** 直方图 (for grouped data, where the **area** of each bar shows the frequency);
- a **cumulative frequency** 累积频数 graph (running totals, used to estimate the median and quartiles).

Choosing a representation. Exams ask you to *select* a diagram and *discuss* its advantages and disadvantages. Match the diagram to the data and to the job:

Diagram	Use it for	Advantage	Disadvantage
stem-and-leaf	small sets of raw data	keeps every original value; shows shape; median and quartiles can be read off	awkward for large data sets
box-and-whisker plot	comparing two or more sets	shows median, quartiles, range and skew at a glance; easy to compare	loses the individual values; hides the shape inside the box (two peaks can look like one)
histogram	large sets of grouped continuous data	shows the shape of the distribution; works with unequal class widths	loses the original values; median and quartiles are not visible
cumulative frequency graph	large grouped data	median, quartiles and percentiles can be estimated; proportion above or below any value	loses the original values; the shape is harder to see

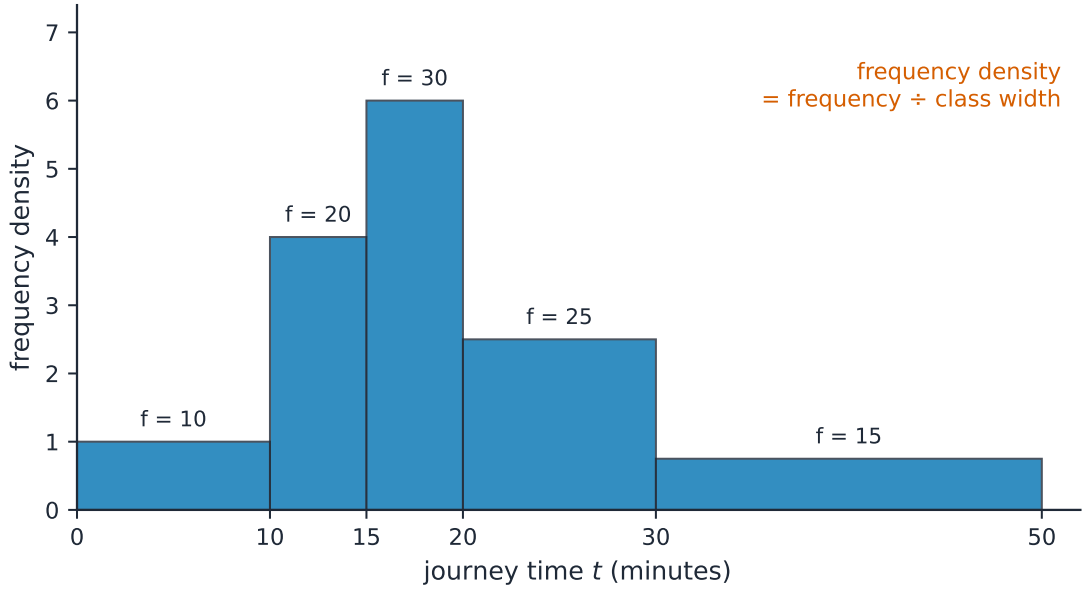
A full "discuss" answer names one advantage **and** one disadvantage, each tied to the data in the question: "A box-and-whisker plot would let the two classes' medians and IQRs be compared directly, but the individual marks would be lost."

Histograms with unequal class widths. The **area** of each bar, not its height, shows the frequency. So the height is the **frequency density** 频数密度:

$$\text{frequency density} = \frac{\text{frequency}}{\text{class width}}.$$

> **Worked example: drawing and reading a histogram.** The journey times t minutes of 100 people are grouped as $0 \leqslant t < 10$: 10; $10 \leqslant t < 15$: 20; $15 \leqslant t < 20$: 30; $20 \leqslant t < 30$: 25; $30 \leqslant t < 50$: 15. Draw a histogram and estimate how many people took between 12 and 25 minutes.

The class widths are $10, 5, 5, 10, 20$, so the frequency densities are $1, 4, 6, 2.5, 0.75$. Draw bars with these heights, with no gaps, from the true class boundaries.



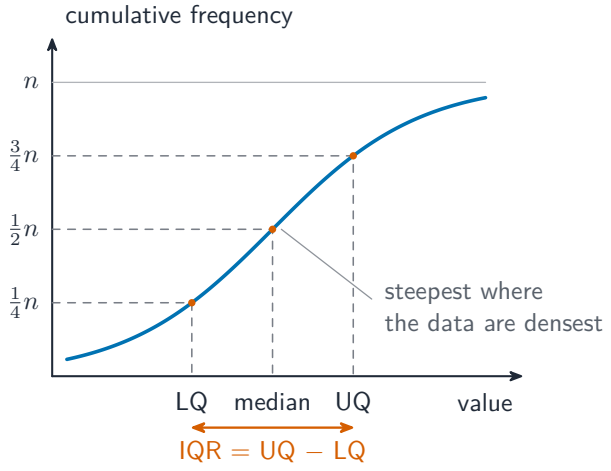
With unequal class widths, bar height is frequency density, so each bar's area is its frequency

For the estimate, take the area between 12 and 25: $(15-12)\times 4+30+(25-20)\times 2.5 = 12+30+12.5 = 54.5$, so about 55 people. This assumes values are spread evenly within each class.

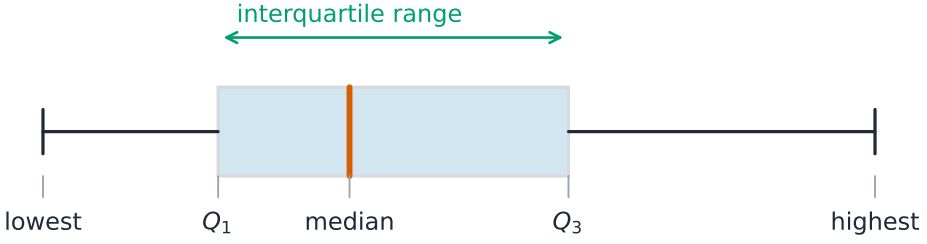
The common errors are plotting frequency instead of frequency density, and leaving gaps at boundaries such as 9.5 for data recorded to the nearest whole number.

Reading a cumulative frequency graph. Plot each cumulative frequency at the **upper class boundary**, then join with a smooth curve (or straight lines). To find the 90th percentile of n values, read across from $0.9n$. To find how many values are **above** x, read the cumulative frequency at x and subtract it from n.

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A cumulative frequency curve never decreases; read the median across from half the total frequency, and the quartiles from one quarter and three quarters of the total



box: Q_1–Q_3 · median line · whiskers: min/max

The box spans the quartiles Q_1 to Q_3; the whiskers reach the lowest and highest values.

■ Averages and spread

A **measure of central tendency** 集中趋势 is a single "middle" value:

- the **mean** 平均数 $\bar{x} = \dfrac{\sum x}{n}$ (the average);
- the **median** 中位数 (the middle ordered value, or the mean of the middle pair for an even number of values);
- the **mode** 众数 (the most common value).

A measure of **variation** 离散程度 shows how spread out the data is:

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- the **range (of data)** 极差 (highest − lowest);
- the **interquartile range** 四分位距 (upper quartile − lower quartile);
- the **standard deviation** 标准差 $\sigma = \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2}$.

You often work from the totals $\sum x$ and $\sum x^2$. The square of the standard deviation is the **variance** 方差.

Choosing a measure. The median and IQR are not affected by extreme values, so use them for skewed data or data with outliers. The mean and standard deviation use every value, so they suit roughly symmetric data. When you **compare** two sets, write one sentence on the average and one on the spread, each in context: "On average group A took longer (median 24 min against 19 min); group A's times were also more varied (IQR 11 against 6)."

Grouped data. For data in classes, use the **mid-interval value** 组中值 x of each class with frequency f:

$$\bar{x} = \frac{\sum fx}{\sum f}, \qquad \sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}.$$

These are **estimates**, because the exact values inside each class are unknown.

Worked example: grouped mean and standard deviation. Use the journey-time data from the histogram above. The mid-interval values are $5, 12.5, 17.5, 25, 40$.

$$\sum fx = 50 + 250 + 525 + 625 + 600 = 2050, \qquad \bar{x} = \frac{2050}{100} = 20.5 \text{ minutes}.$$

$$\sum fx^2 = 250 + 3125 + 9187.5 + 15625 + 24000 = 52187.5,$$

$$\sigma = \sqrt{\frac{52187.5}{100} - 20.5^2} = \sqrt{521.875 - 420.25} = \sqrt{101.625} = 10.1 \text{ minutes}.$$

Worked example. For 10 values, $\sum x = 50$ and $\sum x^2 = 300$. Find the mean and standard deviation.

$$\bar{x} = \frac{50}{10} = 5, \qquad \sigma = \sqrt{\frac{300}{10} - 5^2} = \sqrt{30 - 25} = \sqrt{5} = 2.24.$$

Coding 编码 makes big numbers easier. Replace each value by $t = x - a$ for a convenient **assumed mean** 假定平均数 a. Then $\bar{x} = a + \bar{t}$, while the standard deviation is **unchanged** (shifting every value does not spread the data). So from the coded totals $\sum(x-a)$ and $\sum(x-a)^2$ you get $\bar{x}$ and σ directly, and two data sets can be compared through their coded totals.

Worked example: coded totals. For 8 values, $\sum(x-50) = 24$ and $\sum(x-50)^2 = 200$. Find the mean and standard deviation of x.

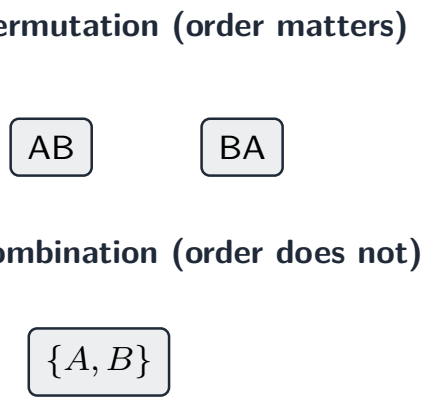
Work with $t = x - 50$ first: $\bar{t} = \dfrac{24}{8} = 3$ and $\sigma_t = \sqrt{\dfrac{200}{8} - 3^2} = \sqrt{16} = 4$. Shift back for the mean only: $\bar{x} = 50 + 3 = 53$, and $\sigma_x = 4$, unchanged. Adding 50 to the standard deviation is the usual error.

Permutations and combinations

A **permutation** 排列 is an arrangement where order matters; a **combination** 组合 is a selection where order does not matter. The numbers are

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!}, \qquad {}^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}. $$

To arrange objects in a line when some are repeated, divide by the factorial of each repeat count.



Order matters for a permutation, but not for a combination

Worked example. How many different arrangements are there of the letters of the word NEEDLESS?

There are 8 letters, with E repeated 3 times and S repeated 2 times:

$$ \frac{8!}{3!\,2!} = \frac{40320}{6 \times 2} = 3360. $$

Worked example: together and apart. Seven friends, including Amy and Bo, stand in a row. In how many ways can they stand if (a) Amy and Bo are together, (b) Amy and Bo are not together?

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(a) Tie Amy and Bo into one block. Now arrange 6 items, $6! = 720$ ways, and the block itself can be AB or BA: $720 \times 2 = 1440$.

(b) Take the total minus the "together" cases: $7! - 1440 = 5040 - 1440 = 3600$.

> **Worked example: "at least one".** A committee of 4 is chosen from 5 men and 4 women. How many committees contain at least one woman?

Count the opposite case: committees with no women use only men, ${}^5C_4 = 5$ ways. All committees: ${}^9C_4 = 126$. So $126 - 5 = 121$.

> **Worked example: not next to each other.** In how many ways can the letters of PARALLEL be arranged so that the three L's are **not** all together, and so that **no two** L's are next to each other?

There are 8 letters: L three times, A twice, P, R and E once each. All arrangements: $\dfrac{8!}{3!\,2!} = 3360$. With the three L's as one block, arrange 6 items with A twice: $\dfrac{6!}{2!} = 360$. So "not all together" is $3360 - 360 = 3000$.

For "no two together", first arrange the other 5 letters (P, A, R, A, E): $\dfrac{5!}{2!} = 60$ ways. They leave 6 gaps (one at each end and four between). Put the L's in 3 different gaps: ${}^6C_3 = 20$ ways. So $60 \times 20 = 1200$. This **gaps method** is the standard way to keep items apart.

Common mistakes. Using ${}^nP_r$ for a committee (order does not matter in a committee), and counting "at least one" case by case, which double-counts easily; the complement is safer.

Arranging 3 of 4 objects ($^4P_3$)



Choosing 3 of 4 (order ignored): ${}^4C_3 = \dfrac{{}^4P_3}{3!} = 4$

${}^nP_r = n!/(n-r)!$ · ${}^nC_r = {}^nP_r/r!$

nPr = n!/(n−r)!; nCr = nPr/r!

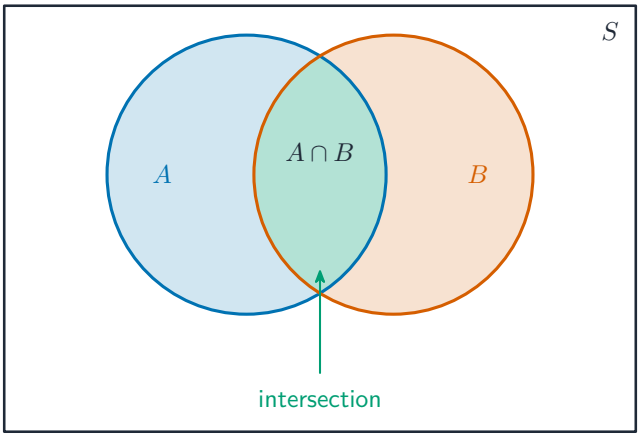
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Dice: a starting point for probability

Find a probability by counting equally likely outcomes, or by using permutations and combinations. Combine probabilities with these rules:

- **addition** for "or": $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;
- **multiplication** for "and" when events are independent: $P(A \cap B) = P(A)\,P(B)$.



The overlap of the two circles is $A \cap B$; the addition rule subtracts it once so it is not counted twice.

Two events are **mutually exclusive events** 互斥事件 if they cannot both happen, and **independent events** 独立事件 if one happening does not change the chance of the other. To test independence, check whether $P(A \cap B) = P(A) \times P(B)$. A **conditional probability** 条件概率 is the chance of A given that B has happened:

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$.

> **Worked example.** Events have $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cap B) = 0.2$. Are A and B independent?

Test: $P(A) \times P(B) = 0.5 \times 0.4 = 0.2 = P(A \cap B)$. The values are equal, so A and B are independent.

> **Worked example: a tree diagram.** The probability that it rains on a given day is 0.3. If it rains, the probability that Kim is late is 0.4; if it does not rain, it is 0.1. Find the probability that Kim is late, and the probability that it rained given that Kim was late.

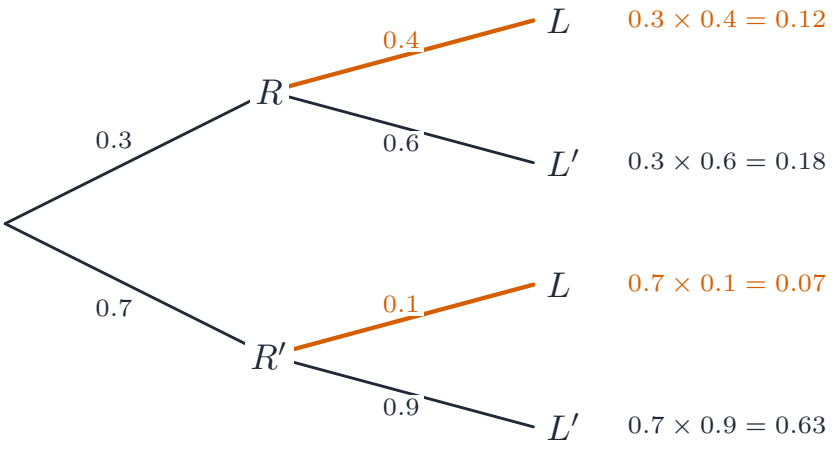
Multiply along each branch that ends in "late", then add the branches:

$$P(L) = 0.3 \times 0.4 + 0.7 \times 0.1 = 0.12 + 0.07 = 0.19.$$

Given that Kim was late, only those two branches count:

$$P(R \mid L) = \frac{P(R \cap L)}{P(L)} = \frac{0.12}{0.19} = 0.632.$$

The common error is to answer 0.4, which is $P(L \mid R)$, the other way round.



$$P(L) = 0.12 + 0.07 = 0.19,$$
$$\text{and } P(R \mid L) = \frac{0.12}{0.19} = 0.632$$

Multiply along each branch; add the branches that end in L.

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Discrete random variables

A **discrete random variable** 离散型随机变量 X takes separate values, each with a probability. List them in a **probability distribution table** 概率分布表; the probabilities must add to 1. Then the **expectation** 期望 (mean) and **variance** are

$$E(X) = \sum x\,P(X = x), \qquad \mathrm{Var}(X) = \sum x^2\,P(X = x) - \big(E(X)\big)^2.$$

Two special models:

- the **binomial distribution** 二项分布 $X \sim B(n, p)$, for the number of successes in n independent trials, each with the same success probability p: $P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$, with $E(X) = np$ and $\mathrm{Var}(X) = np(1-p)$;
- the **geometric distribution** 几何分布, for the trial on which the first success happens in independent trials with constant success probability p: $P(X = r) = (1-p)^{r-1}p$, with $E(X) = \dfrac{1}{p}$.

Choosing the model. Before using a formula, check the conditions in context. Binomial $B(n, p)$: a **fixed number** n of trials; each trial has **two outcomes**; trials are **independent**; the probability p is **constant**. Geometric $\mathrm{Geo}(p)$: the same, but there is no fixed number of trials and X counts the trial on which the **first** success happens. For example, "the number of sixes in 20 throws of a die" is $B(20, \frac{1}{6})$; "the number of throws until the first six" is $\mathrm{Geo}(\frac{1}{6})$. Picking cards **without replacement** is not binomial, because p changes.

Worked example: a probability table. The variable X has this distribution. Find k, $E(X)$ and $\mathrm{Var}(X)$.

x	1	2	3	4
$P(X = x)$	0.1	0.3	k	0.2

The probabilities add to 1, so $k = 1 - 0.6 = 0.4$. Then

$$E(X) = 1(0.1) + 2(0.3) + 3(0.4) + 4(0.2) = 2.7,$$

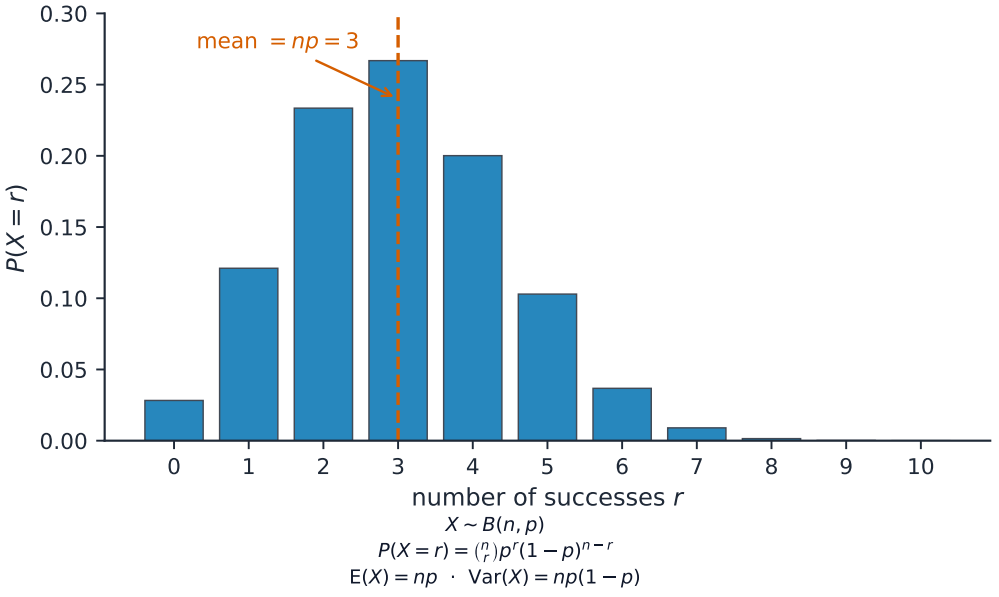
$$E(X^2) = 1(0.1) + 4(0.3) + 9(0.4) + 16(0.2) = 8.1, \qquad \mathrm{Var}(X) = 8.1 - 2.7^2 = 0.81.$$

Remember to subtract $\big(E(X)\big)^2$: stopping at 8.1 is a frequent slip.

Worked example. $X \sim B(10, 0.3)$. Find $P(X = 2)$ and $E(X)$.

$$P(X = 2) = \binom{10}{2}(0.3)^2(0.7)^8 = 45 \times 0.09 \times 0.05765 = 0.233, \qquad E(X) = 10 \times 0.3 = 3.$$

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The distribution $B(10, 0.3)$: each bar is $P(X = r)$, clustered around the mean $np = 3$.

> **Worked example: geometric.** A fair die is thrown until a six appears. Find the probability that the first six comes on the third throw, and that more than 4 throws are needed.

$P(X = 3) = \left(\frac{5}{6}\right)^2 \frac{1}{6} = 0.116$. "More than 4 throws" means the first four throws are all not six: $P(X > 4) = \left(\frac{5}{6}\right)^4 = 0.482$.

Two shortcuts follow: $P(X > r) = (1-p)^r$ and $P(X \leqslant r) = 1 - (1-p)^r$.

> **Worked example: "at least" in a binomial.** $X \sim B(8, 0.25)$. Find $P(X \geqslant 2)$.

Use the complement, so you need only two terms:

$$P(X \geqslant 2) = 1 - P(X = 0) - P(X = 1) = 1 - 0.75^8 - 8(0.25)(0.75)^7 = 1 - 0.1001 - 0.2670 = 0.633.$$

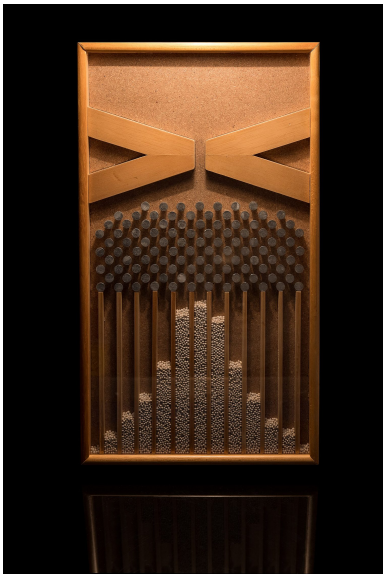
The normal distribution

The **normal distribution** 正态分布 models a **continuous random variable** 连续型随机变量 with a symmetric bell shape. Write $X \sim N(\mu, \sigma^2)$, where μ is the mean and σ is the standard deviation. To use the tables, by **standardisation** 标准化 convert to the variable $Z \sim N(0, 1)$:

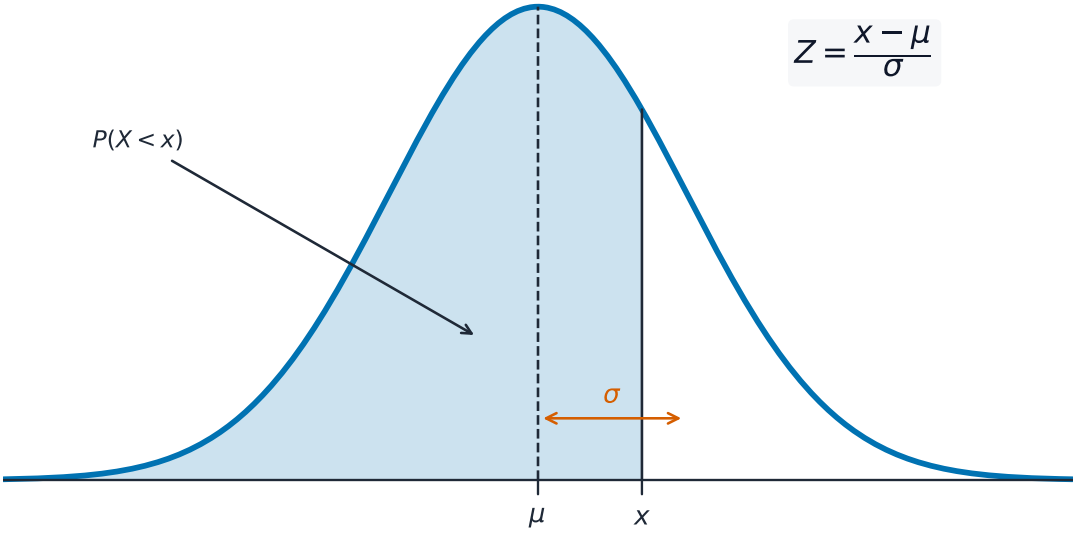
$$Z = \frac{X - \mu}{\sigma}.$$

Then $P(X < x) = P\left(Z < \dfrac{x - \mu}{\sigma}\right)$, which you read from the normal table Φ.

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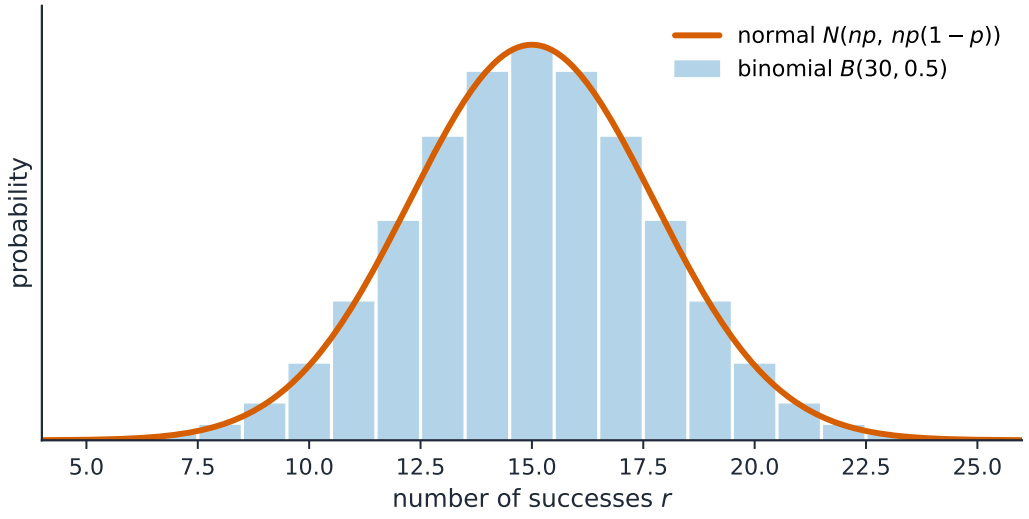


A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.



A normal probability is an area under the bell curve; standardizing rescales it to $Z \sim N(0, 1)$.

The normal distribution is also a good **approximation** 近似 to the binomial when both np and $n(1-p)$ are sufficiently large. Because you replace a discrete variable by a continuous one, apply a **continuity correction** 连续性校正 (adjust by 0.5).



approx $N(np,\ np(1-p))$ when n large · continuity correction

When both np and $n(1-p)$ are sufficiently large, a normal curve can approximate the binomial with the same mean and variance.

Worked example. Bags of rice have mass $X \sim N(\mu, 0.14^2)$. Given that $P(X < 1.48) = 0.22$, find μ.

From the table, $P(Z < z) = 0.22$ gives $z = -0.772$. So

$$\frac{1.48 - \mu}{0.14} = -0.772 \;\Rightarrow\; \mu = 1.48 + 0.772 \times 0.14 = 1.59 \text{ kg (3 s.f.)}.$$

Worked example: finding μ and σ together. The heights of a type of plant are normally distributed. 10% are taller than 45 cm and 20% are shorter than 30 cm. Find μ and σ.

Sketch the curve first: 45 is above the mean and 30 is below it. From the table, $P(Z > 1.282) = 0.10$ and $P(Z < -0.842) = 0.20$. So

$$\frac{45 - \mu}{\sigma} = 1.282, \qquad \frac{30 - \mu}{\sigma} = -0.842.$$

Multiply out: $45 - \mu = 1.282\sigma$ and $30 - \mu = -0.842\sigma$. Subtract: $15 = 2.124\sigma$, so $\sigma = 7.06$ cm and $\mu = 45 - 1.282 \times 7.06 = 35.9$ cm. A z-value for a value below the mean must be **negative**; dropping the sign is the usual slip.

Worked example: a symmetric interval and an expected number. $X \sim N(50, 4^2)$. Find $P(46 < X < 54)$, and the number of values in this interval expected in a sample of 300.

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Both ends are one standard deviation from the mean, so $P(-1 < Z < 1) = 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6826$. The expected number is $300 \times 0.6826 = 204.8$, about 205.

> **Worked example: approximating a binomial.** 40% of a town's residents cycle to work. In a random sample of 100 residents, find the probability that at least 45 cycle to work.

$X \sim B(100, 0.4)$. Here $np = 40$ and $n(1-p) = 60$ are both more than 5, so use $N(np, np(1-p)) = N(40, 24)$. "At least 45" includes 45, so the continuity correction starts at 44.5:

$$P(X \geqslant 45) \approx P\left(Z > \frac{44.5 - 40}{\sqrt{24}}\right) = P(Z > 0.919) = 1 - 0.821 = 0.179.$$

For the boundary, picture the bar for 45: it runs from 44.5 to 45.5, and "at least 45" keeps the whole bar. So $P(X > 45)$ would start at 45.5, and $P(X \leqslant 45)$ would end at 45.5.

Exam tips

- Decide whether order matters: **permutations** (${}^nP_r$) when it does, **combinations** (${}^nC_r$) when it does not.
- For a **discrete random variable**, check the probabilities sum to 1 and use $E(X) = \sum x\,P(X = x)$.
- For the **normal distribution**, standardise with $z = (x-\mu)/\sigma$, **sketch and shade**, then read the table.
- Apply a **continuity correction** when approximating a discrete variable by the normal.
- For a histogram, plot **frequency density**; for a cumulative frequency graph, plot at the **upper class boundary**.
- When asked which model fits, state the binomial conditions **in context** (fixed n, independent trials, constant p).
- With two unknowns in a normal question, write two standardisation equations and solve them simultaneously; check each z has the right sign from a sketch.

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Probability & Statistics 2

A-Level Mathematics

This handout covers Topic 6: **Probability & Statistics** 概率统计 2. It adds the Poisson model, combining random variables, continuous distributions, and the ideas of estimation and testing.

The Poisson distribution



A Poisson model can describe independent arrivals at a steady average rate.

The **Poisson distribution** 泊松分布 $X \sim \text{Po}(\lambda)$ models the number of random events in a fixed interval, when events occur independently at a steady average rate, with mean λ in that interval:

$$P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!}.$$

For a Poisson variable the mean and the variance are both equal to λ. The Poisson distribution is a good **approximation** 近似 to the **binomial distribution** when n is large and p is small. The normal distribution (with continuity correction) approximates the Poisson when λ is large.

Worked example. $X \sim \text{Po}(3)$. Find $P(X = 2)$.

$$P(X = 2) = e^{-3}\frac{3^2}{2!} = e^{-3} \times 4.5 = 0.224.$$

Worked example: changing the interval. Calls reach a help desk at random, at an average rate of 3 per hour. Find the probability of at least 2 calls in a 2-hour period.

Scale the rate to the interval first: in 2 hours the mean is $2 \times 3 = 6$, so $X \sim \text{Po}(6)$. "At least 2" is easiest from the complement:

$$P(X \geqslant 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-6}(1 + 6) = 1 - 7e^{-6} = 0.983.$$

Worked example: approximating a binomial. 1.5% of items from a machine are faulty. Use a suitable approximation to find the probability that a box of 200 contains at most 1 faulty item.

$X \sim B(200, 0.015)$, with n large and p small, so use $\text{Po}(np) = \text{Po}(3)$:

$$P(X \leqslant 1) \approx e^{-3}(1 + 3) = 4e^{-3} = 0.199.$$

(The exact binomial value is 0.197, so the approximation is good.)

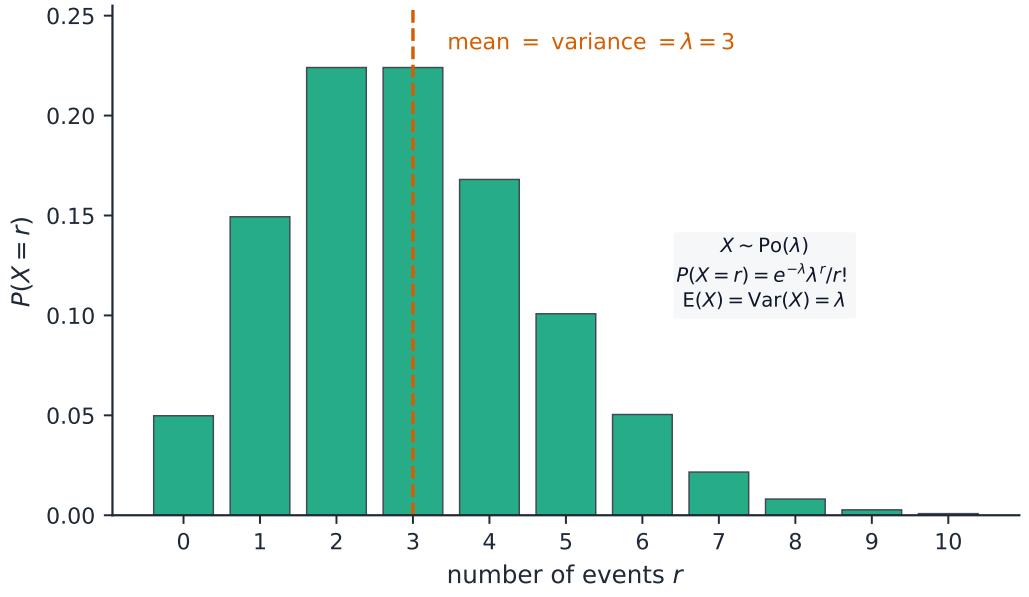
Worked example: a normal approximation. Emails arrive at random at an average of 40 per day. Use a suitable approximation to find the probability of at least 50 emails in a day.

$X \sim \text{Po}(40)$. λ is large, so use $Y \sim N(40, 40)$: the mean and the variance are both λ. X takes whole-number values, so apply a **continuity correction** 连续性校正: "at least 50" becomes $Y > 49.5$.

$$P(Y > 49.5) = P\left(Z > \frac{49.5 - 40}{\sqrt{40}}\right) = P(Z > 1.502) = 0.0665.$$

Use $\sqrt{40}$, the standard deviation, in the denominator, not the variance 40.

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The Poisson distribution Po(3)*: for a Poisson variable the mean and variance both equal* λ*.*

Linear combinations of random variables

When you change a variable by a linear rule, the **expectation** 期望 (mean) and **variance** 方差 follow these rules:

$$E(aX+b) = aE(X)+b, \qquad \operatorname{Var}(aX+b) = a^2 \operatorname{Var}(X).$$

For two **independent** variables X and Y:

$$E(aX+bY) = aE(X)+bE(Y), \qquad \operatorname{Var}(aX+bY) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y).$$

Two useful facts: if X has a **normal distribution** 正态分布 then so does $aX+b$ for $a \neq 0$; and the sum of independent Poisson variables is again Poisson.

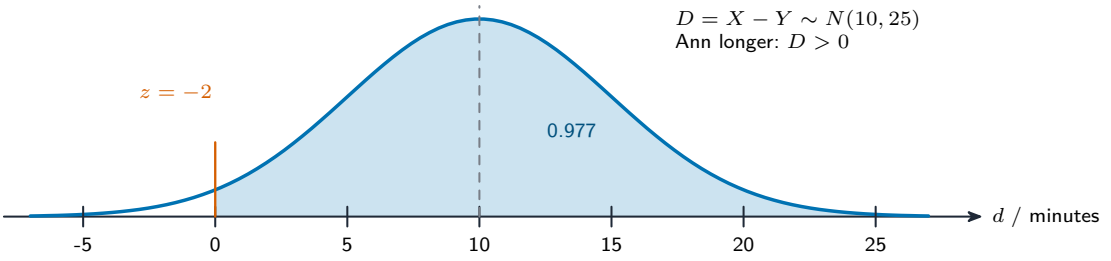
Worked example. X has mean 5 and variance 4. Find $E(3X-1)$ and $\operatorname{Var}(3X-1)$.

$$E(3X-1) = 3(5)-1 = 14, \qquad \operatorname{Var}(3X-1) = 3^2 \times 4 = 36.$$

Worked example: a difference. The time for Ann to finish a task is $X \sim N(70, 16)$ and for Ben is $Y \sim N(60, 9)$, in minutes, independently. Find the probability that Ann takes longer than Ben.

Let $D = X - Y$. Then $E(D) = 70 - 60 = 10$ and $\operatorname{Var}(D) = 16 + 9 = 25$ (the variances **add**, even for a difference). So $D \sim N(10, 25)$ and

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Ann takes longer whenever D is above 0: almost all of the curve.

$$P(D > 0) = P\left(Z > \frac{0 - 10}{5}\right) = P(Z > -2) = 0.977.$$

Worked example: adding Poisson variables. A team scores goals at home at random with mean 1.4 per match, and its opponents score independently with mean 1.1. Find the probability that a match has at most 2 goals in total.

The total T of two independent Poisson variables is Poisson with the means added: $T \sim \text{Po}(2.5)$.

$$P(T \leqslant 2) = e^{-2.5}\left(1 + 2.5 + \frac{2.5^2}{2}\right) = 6.625\, e^{-2.5} = 0.544.$$

Common mistake. $\text{Var}(X_1 + X_2) = 2\,\text{Var}(X)$ for two independent items with the same variance, but $\text{Var}(2X) = 4\,\text{Var}(X)$ for one item doubled. Read which one the question describes.

Continuous random variables

A **continuous random variable** 连续型随机变量 can take any value in a range. Its probabilities come from a **probability density function** 概率密度函数 $f(x)$, with two key properties:

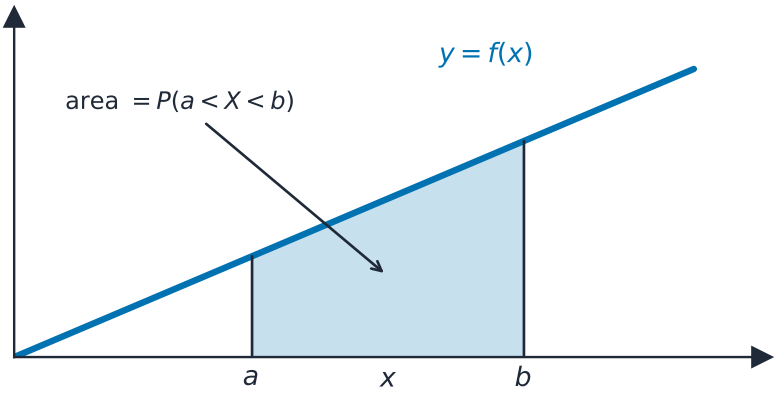
$$f(x) \geqslant 0, \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

A probability is the area under f, and the mean is found by integration:

$$P(a < X < b) = \int_a^b f(x)\,dx, \qquad E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx.$$

The **cumulative distribution function** 累积分布函数 is $F(x) = P(X \leqslant x) = \int_{-\infty}^{x} f(t)\,dt$; the **median** 中位数 solves $F(m) = 0.5$, and other **percentiles** 百分位数 solve $F(x) = p$.

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$$\int_{-\infty}^{\infty} f = 1 \;\cdot\; P(a < X < b) = \int_a^b f$$

For a continuous variable, the probability $P(a < X < b)$ is the area under $f(x)$ between a and b.

Worked example: finding a constant. X has $f(x) = k(4 - x^2)$ for $0 \leqslant x \leqslant 2$, and 0 elsewhere. Find k and $P(X < 1)$.

The total area must be 1:

$$\int_0^2 k(4 - x^2)\,dx = k\left[4x - \tfrac{1}{3}x^3\right]_0^2 = k\left(8 - \tfrac{8}{3}\right) = \tfrac{16}{3}k = 1 \;\Rightarrow\; k = \tfrac{3}{16}.$$

Then $P(X < 1) = \frac{3}{16}\left[4x - \frac{1}{3}x^3\right]_0^1 = \frac{3}{16} \cdot \frac{11}{3} = \frac{11}{16}$. Check that $f(x) \geqslant 0$ on the whole interval: here $4 - x^2 \geqslant 0$ for $0 \leqslant x \leqslant 2$.

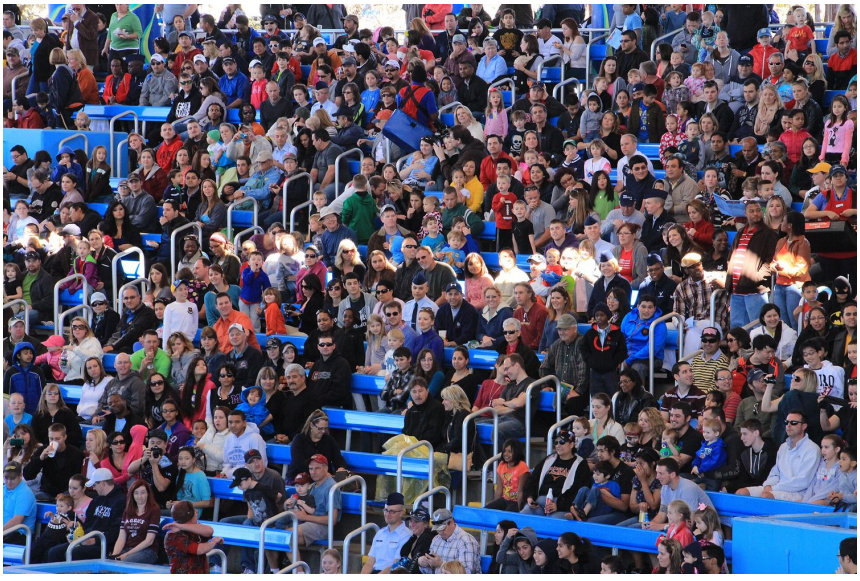
Worked example. A continuous variable has $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 2$ (and 0 elsewhere). Find $E(X)$.

$$E(X) = \int_0^2 x \cdot \tfrac{1}{2}x\,dx = \int_0^2 \tfrac{1}{2}x^2\,dx = \left[\tfrac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}.$$

The **variance** uses the same idea, $\mathrm{Var}(X) = \displaystyle\int_{-\infty}^{\infty} x^2 f(x)\,dx - \big(E(X)\big)^2$. For the same $f(x) = \frac{1}{2}x$ on $[0, 2]$: $\displaystyle\int_0^2 x^2 \cdot \tfrac{1}{2}x\,dx = \left[\tfrac{x^4}{8}\right]_0^2 = 2$, so $\mathrm{Var}(X) = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9}$.

For the **median** of the same variable, find $F(x) = \displaystyle\int_0^x \tfrac{1}{2}t\,dt = \tfrac{1}{4}x^2$ and solve $\frac{1}{4}m^2 = 0.5$: $m = \sqrt{2} = 1.41$. It is above the mean (1.33) because more of the area lies towards $x = 2$.

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Statistics studies a sample to learn about a whole population.

A **sample** 样本 is a small group chosen from the whole **population** 总体. A **random sample** needs **randomness** 随机性, so that every member has a fair chance of being chosen.

Some methods are **unsatisfactory** because they are **biased** 有偏: sampling only volunteers, or the first 20 people to arrive, over-represents certain kinds of people. A genuinely random sample uses **random numbers** – number every member of the population, then draw numbers (from a table or a generator) to decide who is in the sample.

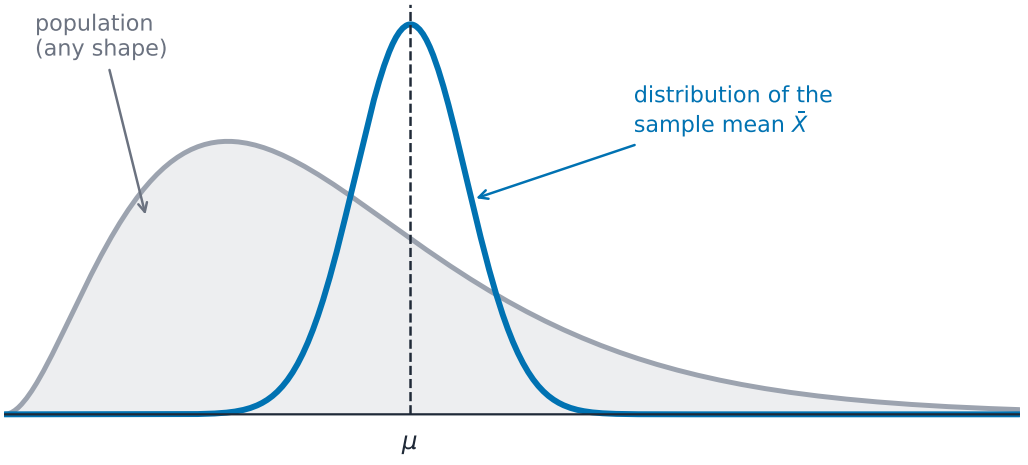
> **Worked example.** To find how long students spend on homework, a school asks the first 30 students who arrive at the library one evening. Explain why this sample is unsatisfactory, and describe a better method. Students who go to the library in the evening are likely to do **more** homework than others, so the sample is biased and not every student had the same chance of being chosen. Instead, number all the students in the school from 1 to N (using the school list), use a random number generator to choose 30 different numbers, ignoring repeats and numbers above N, and ask the students with those numbers.

A sample is used instead of the whole population because a full survey (a **census**) would take too long or cost too much, or because testing destroys the item (such as the lifetime of light bulbs).

For independent observations from a population with finite mean μ and variance σ^2, the sample mean $\bar{X}$ is itself a random variable, with

$$E(\bar{X}) = \mu, \qquad \mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

By the **Central Limit Theorem** 中心极限定理, for a sufficiently large independent sample from the same population with finite, nonzero variance, $\bar{X}$ is approximately normal. The sample size needed depends on the population shape.

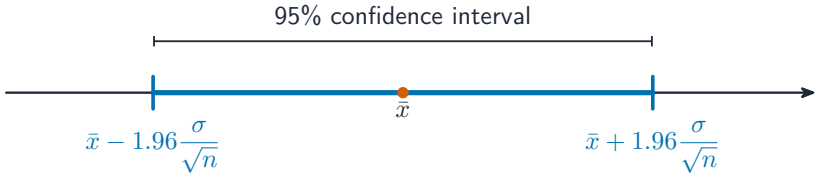


CLT: $\bar{X} \approx N(\mu, \sigma^2/n)$ for large n · SE $= \sigma/\sqrt{n}$

For sufficiently large independent samples from the same population with finite, nonzero variance, $\bar{X}$ is approximately normal, centred on μ.

From a sample you can find **unbiased estimates** 无偏估计 of the population mean and variance. A **confidence interval** 置信区间 uses a method that contains the fixed population mean in the stated fraction of repeated samples. A particular computed interval either contains that mean or does not. When the population is normal with known σ (or the sample is large), a 95% interval is

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}.$$



A 95% confidence interval stretches 1.96 standard errors each side of the sample mean.

You can also find a confidence interval for a **population proportion** 总体比例 from a large sample. With a sample proportion $\hat{p}$, the approximate interval is

$$\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

Worked example: a proportion. In a random sample of 400 voters, 120 support a plan. Find a 95% confidence interval for the proportion of all voters who support it. $\hat{p} = \frac{120}{400} = 0.3$, and

$$0.3 \pm 1.96\sqrt{\frac{0.3 \times 0.7}{400}} = 0.3 \pm 0.0449 \;\Rightarrow\; (0.255,\ 0.345).$$

The value 0.4 is outside the interval, so a claim that 40% support the plan is not supported by this sample.

Worked example. A sample of $n = 64$ has mean $\bar{x} = 50$, from a population with $\sigma = 8$. Find a 95% confidence interval for the population mean.

$$50 \pm 1.96 \times \frac{8}{\sqrt{64}} = 50 \pm 1.96 \;\Rightarrow\; (48.0,\ 52.0).$$

Worked example: choosing the sample size. With $\sigma = 8$, how large must n be for the total width of a 95% interval to be less than 2? The width is $2 \times 1.96 \times \frac{8}{\sqrt{n}} < 2$, so $\sqrt{n} > 15.68$ and $n > 245.9$: take $n = 246$. To halve the width you need **four** times as many observations.

When σ is not known, estimate it from the sample. The unbiased estimate of the population variance divides by $n - 1$, not n:

$$s^2 = \frac{1}{n-1}\left(\sum x^2 - \frac{(\sum x)^2}{n}\right).$$

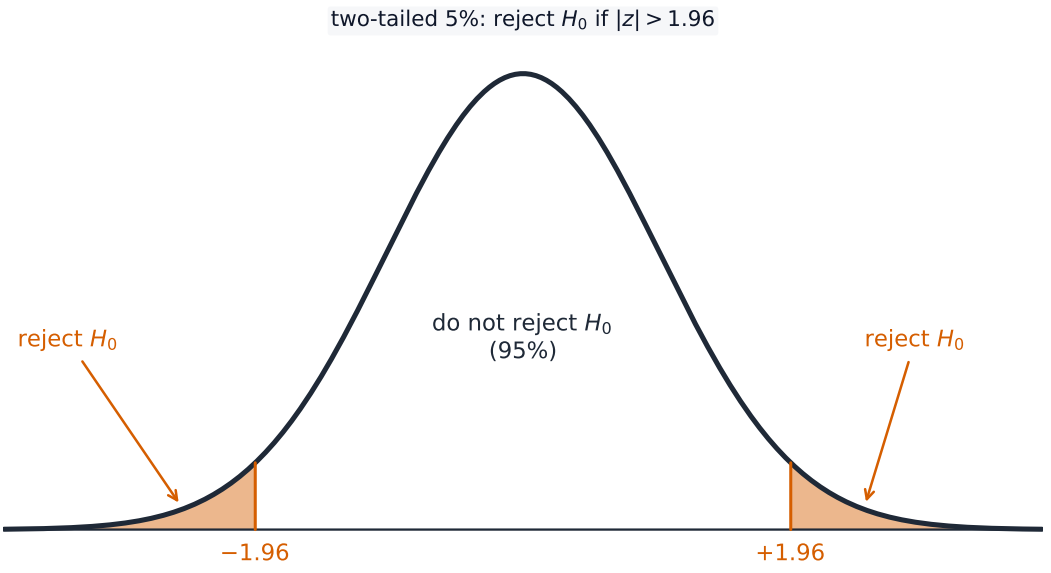
For example, $n = 10$, $\sum x = 50$ and $\sum x^2 = 300$ give $\bar{x} = 5$ and $s^2 = \dfrac{300 - 250}{9} = 5.56$. Dividing by 10 instead would give 5, which is too small on average.

Hypothesis tests

A **hypothesis test** 假设检验 uses sample data to judge a claim. You set up two statements: the **null hypothesis** 原假设 H_0 (the claim being tested, usually "no change") and the **alternative hypothesis** 备择假设 H_1 (what you suspect instead). The test is **one-tailed** 单尾 if H_1 points one way (e.g. $\mu > 50$) and **two-tailed** 双尾 if it allows both ways ($\mu \neq 50$).

You fix a **significance level** 显著性水平 (often 5%), work out a **test statistic** 检验统计量 from the data, and see whether it lands in the **rejection region** 拒绝域 (also called the **critical region**); if it does you reject H_0, otherwise the statistic is in the **acceptance region**, meaning that you fail to reject H_0; this does not prove it true.

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A two-tailed test at 5% rejects H_0 only if the test statistic falls in a shaded tail beyond ± 1.96.

Two mistakes are possible: a **Type I error** 第一类错误 is rejecting H_0 when it is actually true; a **Type II error** 第二类错误 is failing to reject H_0 when it is actually false. You can find their probabilities from the rejection region: $P(\text{Type I}) = P(\text{statistic in the rejection region} \mid H_0)$ – for a normal test this equals the significance level, but for a binomial or Poisson test it is the actual tail probability, which is at most the significance level – and $P(\text{Type II}) = P(\text{statistic in the acceptance region} \mid H_1 \text{ true for a stated value})$, computed from the binomial, Poisson, or normal distribution.

> **Worked example.** A population is claimed to have mean 50, with $\sigma = 8$. A sample of $n = 64$ gives $\bar{x} = 52$. Test at the 5% level whether the mean has changed.

$H_0 : \mu = 50$ and $H_1 : \mu \neq 50$ (two-tailed). The test statistic is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{52 - 50}{8/8} = 2.$$

The critical value at 5% (two-tailed) is 1.96. Since $2 > 1.96$, you reject H_0: there is evidence the mean has changed.

> **Worked example: a test with an estimated variance.** A machine is meant to fill bags with a mean of 25 g. A random sample of 50 bags has $\bar{x} = 24.1$ g and an unbiased variance estimate $s^2 = 9.0$. Test at the 5% level whether the mean is less than 25 g.

$H_0 : \mu = 25$, $H_1 : \mu < 25$ (one-tailed). The sample is large, so by the Central Limit Theorem $\bar{X}$ is approximately normal, and s can be used for σ:

$$z = \frac{24.1 - 25}{3/\sqrt{50}} = -2.12.$$

The critical value is -1.645, and $-2.12 < -1.645$, so reject H_0: there is evidence at the 5% level that the mean mass is less than 25 g. The Central Limit Theorem is needed here because nothing says the masses themselves are normally distributed.

Worked example: a binomial test. A coin is claimed fair but suspected of landing heads too rarely: $H_0 : p = 0.5$, $H_1 : p < 0.5$. In $n = 30$ tosses you see $X = 9$ heads. Under H_0, $X \sim B(30, 0.5)$, so the one-tailed tail probability is

$$P(X \leqslant 9) = \sum_{k=0}^{9} \binom{30}{k} (0.5)^{30} \approx 0.021.$$

Since $0.021 < 0.05$, reject H_0: the coin does seem biased against heads. (For large n the binomial is approximated by a normal; a **Poisson** test works the same way for rare events. And here, if the rule is "reject when $X \leqslant 9$", then $P(\text{Type I}) = P(X \leqslant 9 \mid p = 0.5) \approx 0.021$.)

Worked example: a Poisson test and its errors. A shop usually sells 4 umbrellas a day on average. After an advert, the manager tests at the 5% level whether sales have increased, using one day's sales X. Find the rejection region and the probabilities of Type I and Type II errors, given that the true mean after the advert is 7.

$H_0 : \lambda = 4$ and $H_1 : \lambda > 4$. Under H_0, $X \sim \text{Po}(4)$. Find the smallest c with $P(X \geqslant c) < 0.05$:

$$P(X \geqslant 8) = 0.0511 > 0.05, \qquad P(X \geqslant 9) = 0.0214 < 0.05.$$

So the rejection region is $X \geqslant 9$, and $P(\text{Type I}) = 0.0214$. A Type II error means $X \leqslant 8$ when really $X \sim \text{Po}(7)$: $P(\text{Type II}) = P(X \leqslant 8 \mid \lambda = 7) = 0.729$. With only one day of data, the test misses a real rise most of the time.

Exam tips

- Use the **Poisson** distribution for rare, random, independent events; its mean equals its variance ($= \lambda$).
- When combining independent random variables, **variances add** (they never subtract).
- For a **hypothesis test**, state H_0 and H_1, the significance level, the test statistic, and a conclusion **in context**.
- For a **confidence interval**, use the correct z (or t) value and interpret it in words.

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- Use a **continuity correction** when a normal distribution approximates a binomial or Poisson one: $P(X \geqslant 50)$ becomes $P(Y > 49.5)$.
- When the population is not known to be normal, say that the **Central Limit Theorem** makes the sample mean approximately normal because n is large.
- In a pdf question with a constant k, use the total area $= 1$ first.

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