Course Companion

A-Level Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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Pure Mathematics 1

A-Level Mathematics

This handout covers Topic 1: **Pure Mathematics** 纯数学 1. It is the algebra 代数 and calculus 微积分 core of the course. Each `##` section is one syllabus subtopic.

Quadratics

A suspension bridge: the main cable hangs in a parabola.

A **quadratic** 二次式 is an expression of the form $ax^2 + bx + c$, where $a \neq 0$. The letters a, b, c are the **coefficients** 系数 (the fixed numbers). Much of this section is about solving the **equation** $ax^2 + bx + c = 0$.

Completing the square

To **complete the square** 配方 means to write the quadratic in the form

$$a(x+p)^2 + q.$$

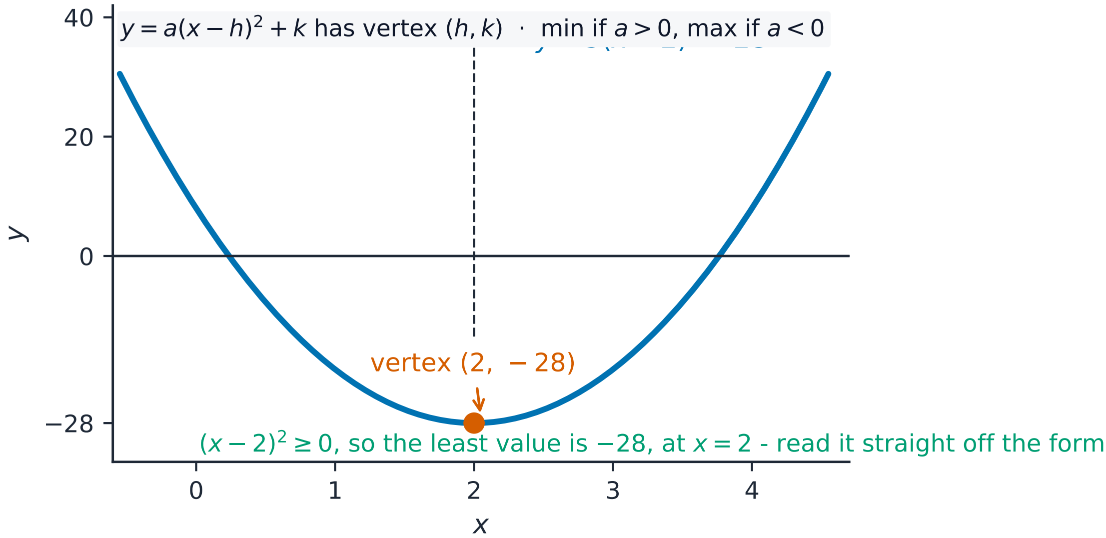
This form is useful: it shows the **vertex** 顶点 (turning point) of the curve at $(-p,\ q)$, and it gives a quick way to solve the equation.

Worked example. Write $9x^2 - 36x + 8$ in the form $p(x+q)^2 + r$.

Take the factor 9 out of the first two terms, then complete the square inside:

$$
\begin{aligned}
9x^2 - 36x + 8 &= 9\left(x^2 - 4x\right) + 8 \\
&= 9\left((x-2)^2 - 4\right) + 8 \\
&= 9(x-2)^2 - 36 + 8 = 9(x-2)^2 - 28.
\end{aligned}
$$

So $p = 9$, $q = -2$, $r = -28$.



The completed square hands you the vertex: least value -28 at x equals 2

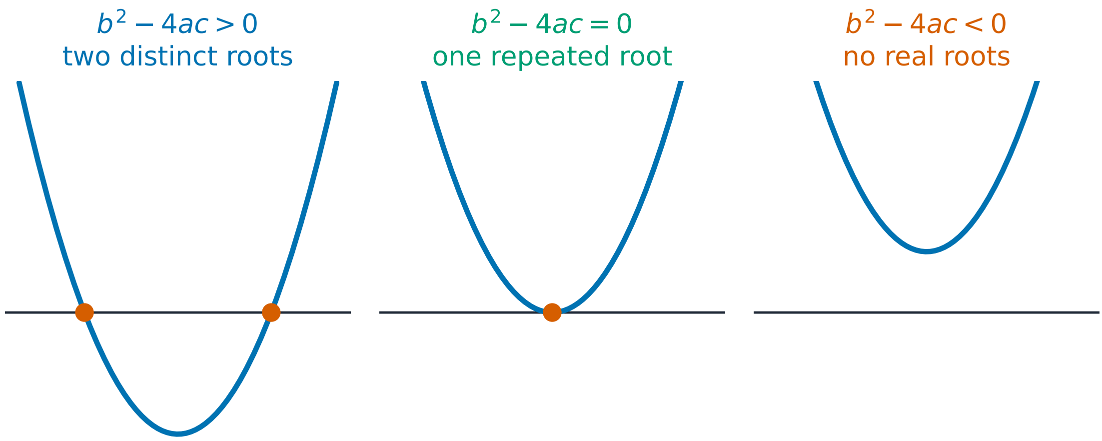
The discriminant

The **discriminant** 判别式 of $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

It tells you how many **real roots** 实根 (real solutions) the equation $ax^2 + bx + c = 0$ has:

Discriminant	Roots
$b^2 - 4ac > 0$	two **distinct** 相异 real roots
$b^2 - 4ac = 0$	one repeated real root
$b^2 - 4ac < 0$	no real roots



$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(real roots need } b^2 - 4ac \geq 0\text{)}$$

The sign of $b^2 - 4ac$ decides how many times the parabola meets the x-axis.

Worked example. Find the values of the constant k for which $3kx^2 + (k+8)x + 3 = 0$ has two distinct real roots.

Here $a = 3k$, $b = k + 8$, $c = 3$. For two distinct real roots you need $b^2 - 4ac > 0$:

$$(k+8)^2 - 4(3k)(3) > 0 \;\Rightarrow\; k^2 + 16k + 64 - 36k > 0 \;\Rightarrow\; k^2 - 20k + 64 > 0.$$

Factorise: $(k - 4)(k - 16) > 0$, so $k < 4$ or $k > 16$. You also need $a \neq 0$, so $k \neq 0$.

Quadratic equations and inequalities

To solve a **quadratic equation** 二次方程, factorise, complete the square, or use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To solve a **quadratic inequality** 二次不等式 such as $(k-4)(k-16) > 0$, find the two roots, then decide which side of each root makes the statement true. A sketch of the parabola 抛物线 helps: the curve is above the x-axis (positive) outside the roots and below it (negative) between them.

Simultaneous equations

To solve a pair of **simultaneous equations** 联立方程 where one is linear and one is quadratic, use **substitution** 代入: rearrange the linear equation for one letter, then put that into the quadratic. This gives a single quadratic to solve.

Equations that are quadratic in disguise

Some equations are quadratic in **some function of** x. For example $x^4 - 5x^2 + 4 = 0$ is quadratic in x^2: let $u = x^2$, solve $u^2 - 5u + 4 = 0$, then go back to x. You will use this idea again in trigonometry.

Functions

A **function** 函数 is a rule that sends each input to exactly one output. Write it as $f(x)$. The **composition of functions** 复合函数 $fg(x)$ means apply g first, then apply f to the result.

- The **domain** 定义域 is the set of allowed inputs x.
- The **range** 值域 is the set of outputs the function actually produces.

A function is **one-one** 一一对应 if different inputs always give different outputs. (No output is repeated.) Only a one-one function has an **inverse function** 反函数 f^{-1}, which reverses the rule.

The **composition** 复合 of two functions means doing one after the other. $fg(x)$ means ”do g first, then f”: $fg(x) = f(g(x))$. The composite function fg exists only when the range of g lies inside the domain of f.

Finding an inverse

To find f^{-1}: write $y = f(x)$, make x the subject, then swap letters.

Worked example. The function $f(x) = (x+3)^2 - 12$ is defined for $x \geqslant 0$. Find $f^{-1}(x)$.

Write $y = (x+3)^2 - 12$ and solve for x:

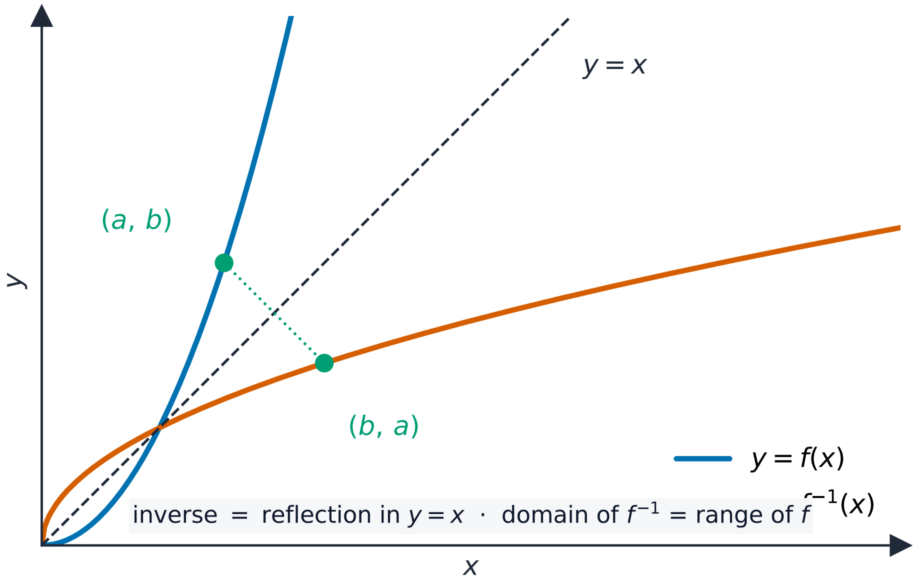
$$(x+3)^2 = y + 12 \;\Rightarrow\; x + 3 = \sqrt{y+12} \;\Rightarrow\; x = \sqrt{y+12} - 3.$$

You take the **positive** square root because $x \geqslant 0$ means $x + 3 \geqslant 3 > 0$. So

$$f^{-1}(x) = \sqrt{x+12} - 3.$$

Graphs of inverses and transformations

The graph of $y = f^{-1}(x)$ is the **reflection** 反射 of $y = f(x)$ in the line $y = x$.

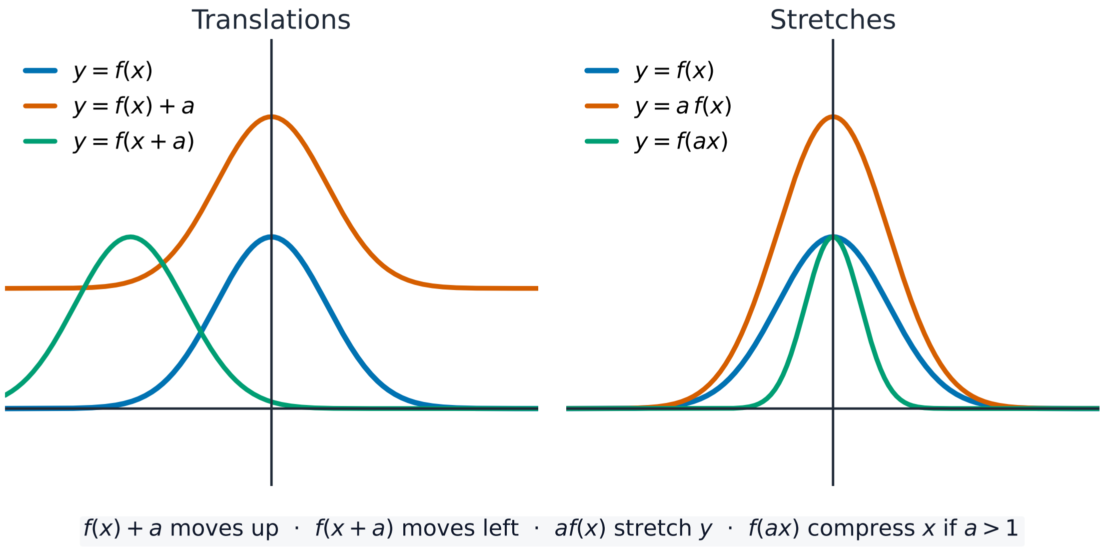


Reflecting $y = f(x)$ in the line $y = x$ gives its inverse; a point (a, b) becomes (b, a).

You should know these **transformations** 变换 of $y = f(x)$:

New equation	Effect on the graph
$y = f(x) + a$	**translation** 平移 up by a
$y = f(x + a)$	translation left by a
$y = a\,f(x)$	**stretch** 伸缩 in the y-direction, scale factor a
$y = f(ax)$	stretch in the x-direction, scale factor $\frac{1}{a}$

When two transformations are combined, the order can matter. State each one fully (type, direction, and amount).



Adding to the output or input slides the curve; a multiplier stretches it.

Coordinate geometry

Coordinate geometry 坐标几何 studies lines and circles using their equations.

Straight lines

The **gradient** 斜率 (steepness) of the line joining (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

You can write the **equation of a straight line** 直线方程 in any of these forms:

$$y = mx + c, \qquad y - y_1 = m(x - x_1), \qquad ax + by + c = 0.$$

Two lines are **parallel** 平行 when their gradients are equal, and **perpendicular** 垂直 (at right angles) when the product of their gradients is -1.

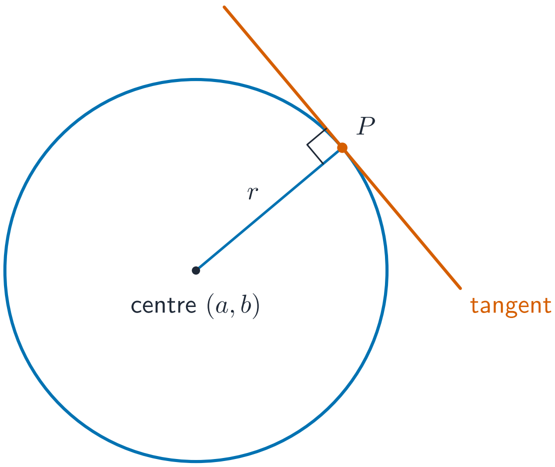
Circles

The **circle** 圆 with **centre** 圆心 (a, b) and **radius** 半径 r has equation

$$(x-a)^2 + (y-b)^2 = r^2.$$

An expanded form like $x^2 + y^2 - 6x + 10y - 27 = 0$ is the same circle: complete the square in x and in y to find the centre and radius.

A **tangent** 切线 to a circle touches it at one point and is perpendicular to the radius at that point. This right-angle fact solves most circle problems.



A tangent touches the circle once and meets the radius at a right angle.

Worked example. The points $P(1, 1)$ and $Q(7, 11)$ are the ends of a **diameter** 直径 of a circle. Find the equation of the circle.

The centre is the midpoint of PQ:

$$\left(\frac{1+7}{2}, \frac{1+11}{2}\right) = (4, 6).$$

The radius is half the length of PQ:

$$r = \tfrac{1}{2}\sqrt{(7-1)^2 + (11-1)^2} = \tfrac{1}{2}\sqrt{36+100} = \tfrac{1}{2}\sqrt{136} = \sqrt{34}.$$

So the circle is $(x-4)^2 + (y-6)^2 = 34$.

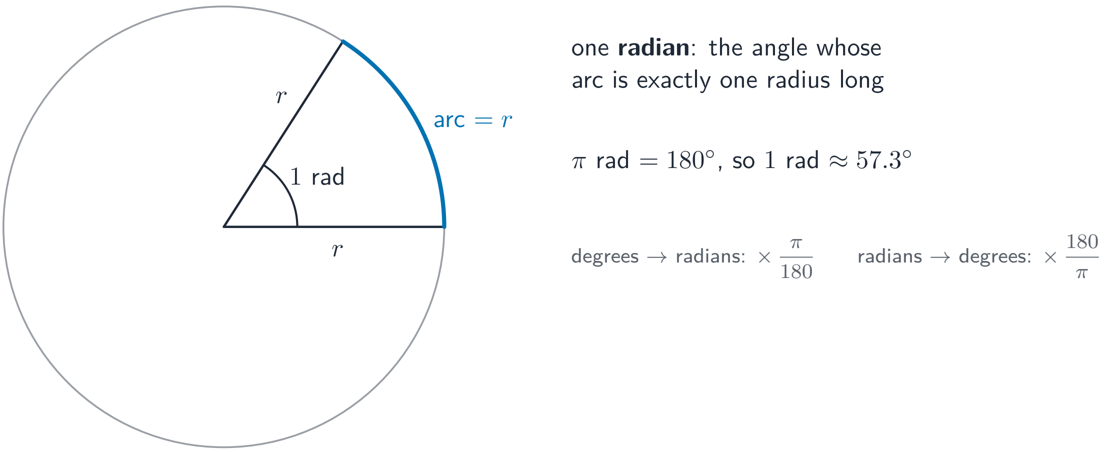
Circular measure

Radians

A **radian** 弧度 is another way to measure angles. One radian is the angle at the centre of a circle that cuts off an **arc** 弧 equal in length to the radius. The link between radians and **degrees** 度 is

$$\pi \text{ radians} = 180°.$$

So to change degrees to radians, multiply by $\dfrac{\pi}{180}$; to change radians to degrees, multiply by $\dfrac{180}{\pi}$.



One radian: the angle whose arc equals the radius

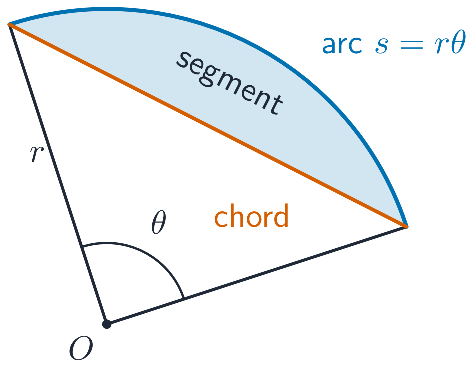
Arc length and sector area

For a **sector** 扇形 with radius r and angle θ **in radians**:

$$\text{arc length} = s = r\theta, \qquad \text{sector area} = A = \tfrac{1}{2}r^2\theta.$$

A **chord** 弦 cuts the sector into a triangle and a **segment** 弓形. The segment area is the sector minus the triangle:

$$\text{segment} = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta = \tfrac{1}{2}r^2(\theta - \sin\theta).$$



The shaded segment is the part of the sector between the chord and the arc.

Worked example. A sector has centre O and the angle at O is $\frac{2}{3}\pi$ radians. Show that the segment cut off by the chord has area about $0.614r^2$.

$$\text{segment} = \tfrac{1}{2}r^2\left(\tfrac{2}{3}\pi - \sin\tfrac{2}{3}\pi\right) = \tfrac{1}{2}r^2(2.0944 - 0.8660) = \tfrac{1}{2}r^2(1.2284) \approx 0.614r^2.$$

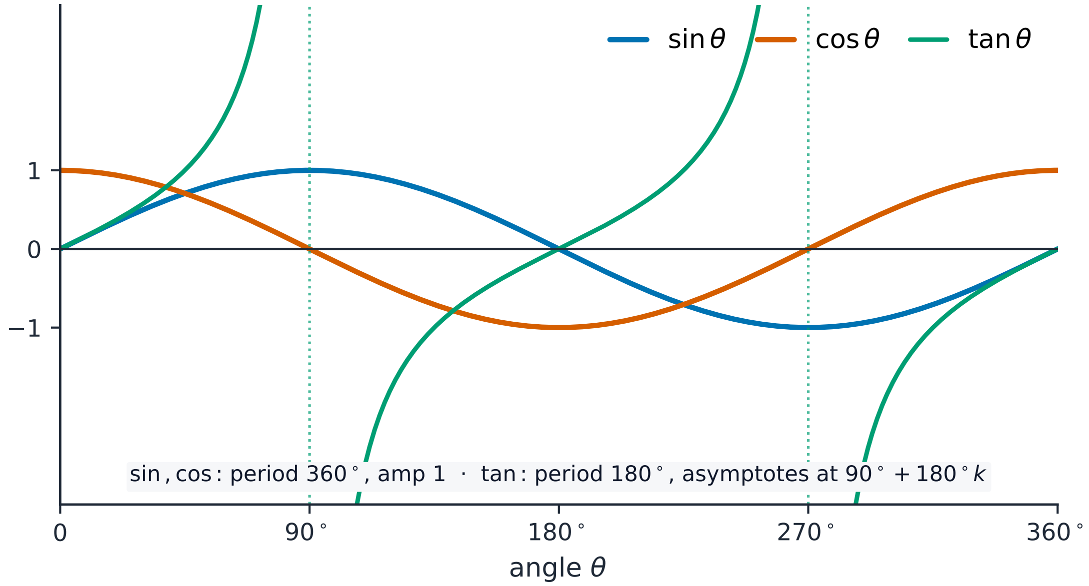
Trigonometry

A Ferris wheel: a point on the rim rises and falls like a sine curve.

Graphs and exact values

You must know the shape of the graphs of the **sine** 正弦, **cosine** 余弦 and **tangent function** 正切 (written sin, cos, tan). The sine and cosine graphs wave between -1 and 1 and repeat every $360°$ (2π). Learn these exact values:

θ	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$



Over one turn sin *and* cos *stay between* -1 *and* 1*;* tan *shoots off at* $90°$ *and* $270°$*.*

The notations $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ mean the **inverse** angle (the **principal value** 主值).

Identities

An **identity** 恒等式 is true for every value of the angle. The two you must know are

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}, \qquad \sin^2\theta + \cos^2\theta \equiv 1.$$

Use them to rewrite an equation so it contains only one trig function.

Solving trigonometric equations

To solve a **trigonometric equation** 三角方程, first reduce it to one function, then find every solution in the given interval.

Worked example. Solve $6\sin\theta = 1 + \dfrac{2}{\sin\theta}$ for $-180° < \theta < 180°$.

Multiply through by $\sin\theta$ to clear the fraction. This makes a **quadratic in** $\sin\theta$:

$$6\sin^2\theta - \sin\theta - 2 = 0 \;\Rightarrow\; (3\sin\theta - 2)(2\sin\theta + 1) = 0.$$

So $\sin\theta = \frac{2}{3}$ or $\sin\theta = -\frac{1}{2}$.

- $\sin\theta = \frac{2}{3}$: $\theta = 41.8°$ or $\theta = 180° - 41.8° = 138.2°$.
- $\sin\theta = -\frac{1}{2}$: $\theta = -30°$ or $\theta = -150°$.

The four solutions are $\theta = -150°,\ -30°,\ 41.8°,\ 138.2°$.

Series

The binomial expansion

For a positive integer n, the **binomial expansion** 二项展开式 is

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n,$$

where $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$ is a **binomial coefficient** 二项式系数.

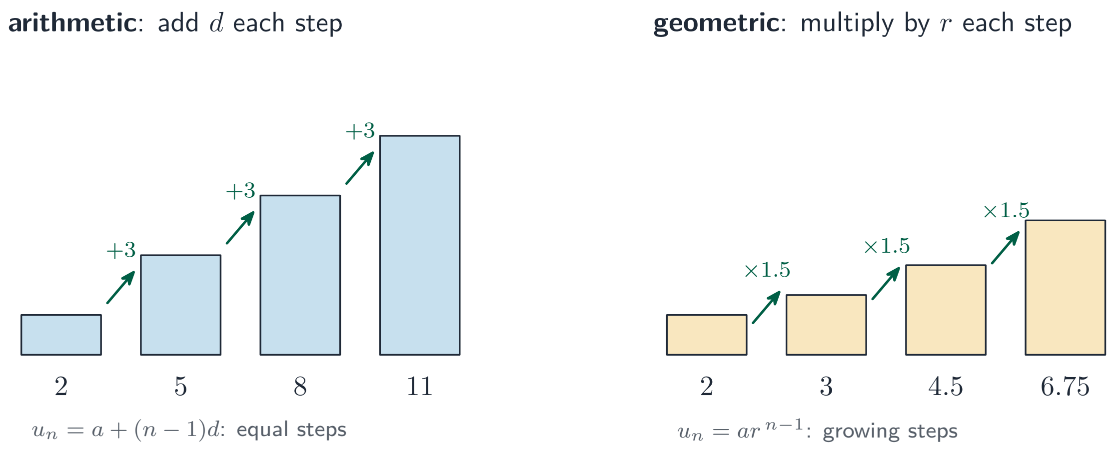
Worked example. Find the first three terms, in ascending powers of x, of $(2 - px)^5$.

$$(2-px)^5 = 2^5 + \binom{5}{1}2^4(-px) + \binom{5}{2}2^3(-px)^2 + \cdots = 32 - 80px + 80p^2x^2 + \cdots$$

Arithmetic and geometric progressions

A **progression** 数列 (sequence) is a list of terms following a rule.

- An **arithmetic progression** 等差数列 (AP) adds a fixed **common difference** 公差 d each step. The nth **term** 项 is $u_n = a + (n-1)d$, and the sum of the first n terms is $S_n = \frac{n}{2}\big(2a + (n-1)d\big)$.
- A **geometric progression** 等比数列 (GP) multiplies by a fixed **common ratio** 公比 r each step. The nth term is $u_n = ar^{n-1}$, and $S_n = \dfrac{a(1-r^n)}{1-r}$.



An AP climbs in equal steps; a GP's steps grow by the same ratio

A GP is **convergent** —it **converges** 收敛 (settles to a limit) —when $|r| < 1$. Then it has a **sum to infinity** 无穷和

$$S_\infty = \frac{a}{1-r}.$$

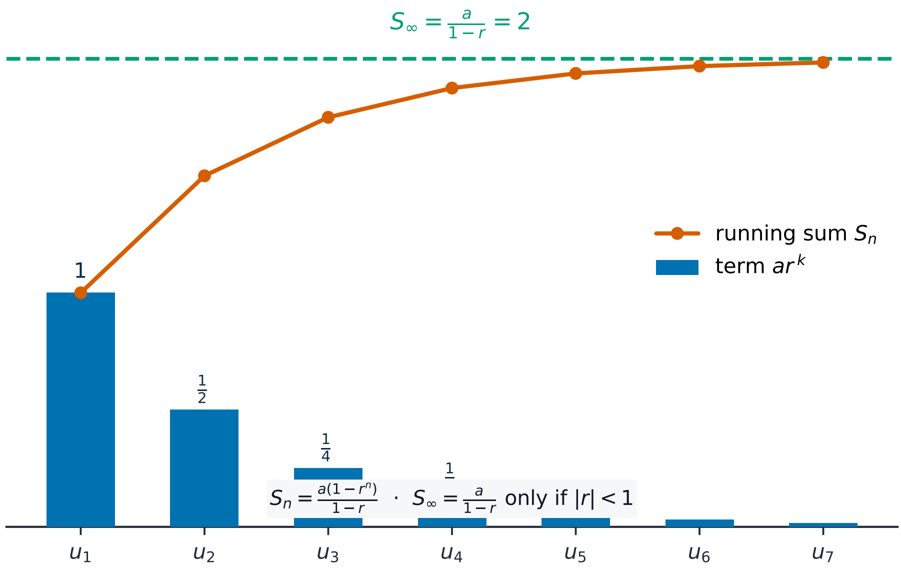
Worked example. The third term of a GP is 18 and the sum of the first three terms is 26. The common ratio is negative. Find the sum to infinity.

From $ar^2 = 18$ you get $a = \dfrac{18}{r^2}$. Put this into $a(1 + r + r^2) = 26$:

$$18(1 + r + r^2) = 26r^2 \;\Rightarrow\; 8r^2 - 18r - 18 = 0 \;\Rightarrow\; (4r+3)(r-3) = 0.$$

The ratio is negative, so $r = -\frac{3}{4}$ and $a = \dfrac{18}{(3/4)^2} = 32$. Then

$$S_\infty = \frac{32}{1-(-\frac{3}{4})} = \frac{32}{\frac{7}{4}} = \frac{128}{7}.$$



Convergent GP: bars shrink toward zero while the running sum approaches S infinity = a/(1-r)

Differentiation

Differentiation 微分 finds the **gradient of a curve** 曲线斜率 at each point. The gradient is the **limit** 极限 of the gradients of shorter and shorter **chords**, called the **derivative** 导数.

The rules

Write the derivative as $f'(x)$ or $\dfrac{dy}{dx}$. The basic rule is

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \quad \text{for any rational } n.$$

Differentiate sums term by term, and use the **chain rule** 链式法则 for a function inside a function:

$$\frac{d}{dx}\, f(g(x)) = f'(g(x)) \cdot g'(x).$$

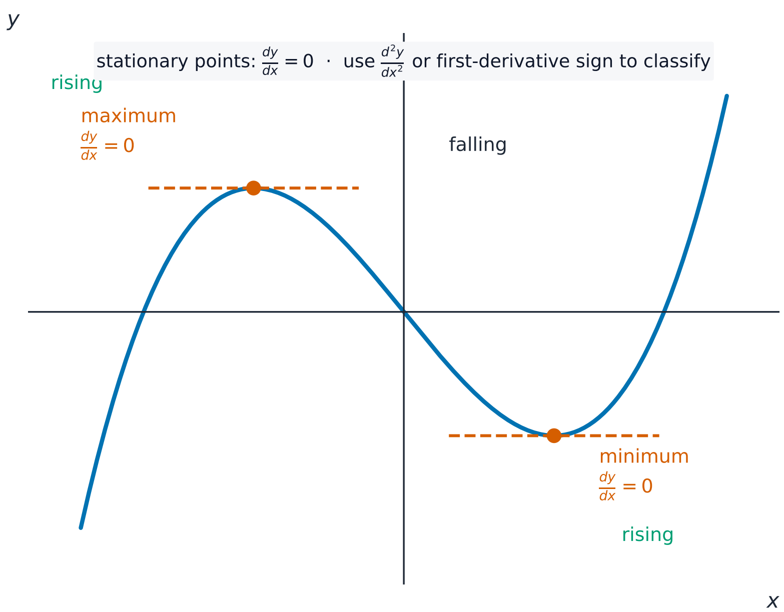
Differentiating again gives the **second derivative** 二阶导数 $f''(x)$ or $\dfrac{d^2y}{dx^2}$.

Using the derivative

- **Tangent and normal.** The gradient of the curve at a point is the gradient of the **tangent** there. The **normal** 法线 is perpendicular to the tangent, so its gradient is $-\dfrac{1}{(\text{tangent gradient})}$.
- **Increasing or decreasing.** The function is an **increasing function** 增函数 where $\dfrac{dy}{dx} > 0$, and a **decreasing function** 减函数 where $\dfrac{dy}{dx} < 0$.
- **Rate of change.** A derivative is a **rate of change** 变化率. Linked rates use the chain rule, e.g. $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$.

Stationary points

A **stationary point** 驻点 is where $\dfrac{dy}{dx} = 0$. Test its nature with the second derivative: $f''(x) > 0$ gives a **minimum point** 极小值点, and $f''(x) < 0$ gives a **maximum point** 极大值点. When $f''(x) = 0$ the test is inconclusive: the point may be a **point of inflexion** 拐点, where the curve changes concavity (the way it bends) –check the sign of $\dfrac{dy}{dx}$ just before and after to decide.



At a maximum or a minimum the tangent is flat, so $\frac{dy}{dx} = 0$.

Worked example. The curve $y = 4x^{1/2} - x$ has a maximum point at $x = a$. Find a.

$$\frac{dy}{dx} = 2x^{-1/2} - 1 = 0 \;\Rightarrow\; \frac{2}{\sqrt{x}} = 1 \;\Rightarrow\; \sqrt{x} = 2 \;\Rightarrow\; x = 4.$$

So $a = 4$.

Integration

Integration 积分 is the reverse of differentiation. Reversing the power rule gives

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad (n \neq -1).$$

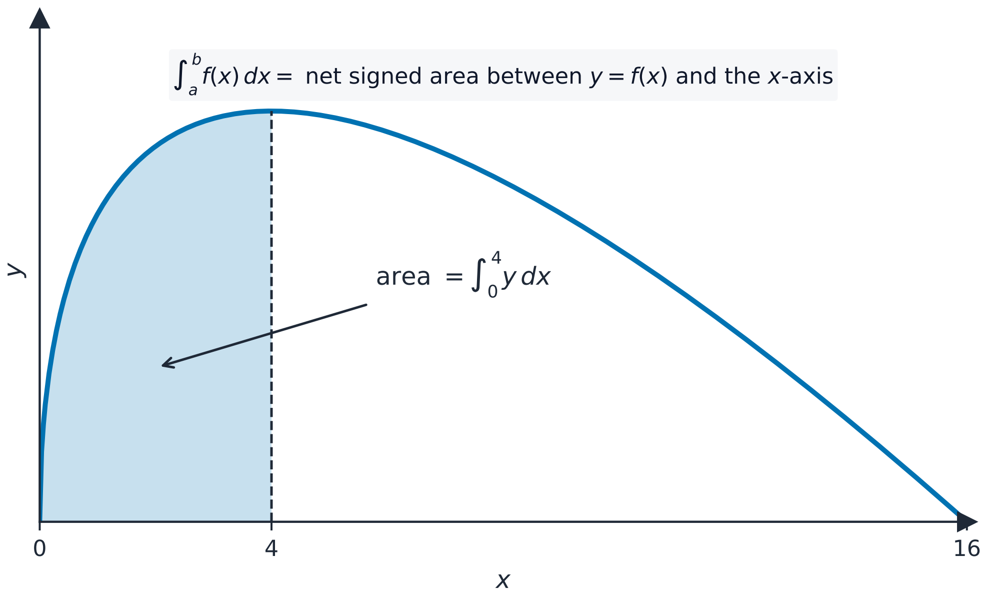
The $+C$ is the **constant of integration** 积分常数. If you know one point on the curve, substitute it to find C.

Definite integrals and area

A **definite integral** 定积分 has limits and gives a number:

$$\int_p^q f(x)\, dx = \big[F(x)\big]_p^q = F(q) - F(p).$$

The area of the **region** 区域 between a curve and the x-axis, from $x = p$ to $x = q$, is $\displaystyle\int_p^q y\, dx$. For the area between two curves, integrate (top curve − bottom curve).



The definite integral $\int_0^4 y\, dx$ is the shaded area under the curve.

Worked example. The curve $y = 4x^{1/2} - x$ meets the x-axis again at $x = 16$. Find the area between the curve and the x-axis from $x = 0$ to $x = 4$.

$$\int_0^4 \left(4x^{1/2} - x\right) dx = \left[\frac{8}{3}x^{3/2} - \frac{x^2}{2}\right]_0^4 = \frac{8}{3}(8) - \frac{16}{2} = \frac{64}{3} - 8 = \frac{40}{3}.$$

An **improper integral** 广义积分 has an infinite limit or an endpoint where the integrand is undefined; evaluate it as a **limit**. For example,

$$\int_1^\infty x^{-2}\, dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1,$$

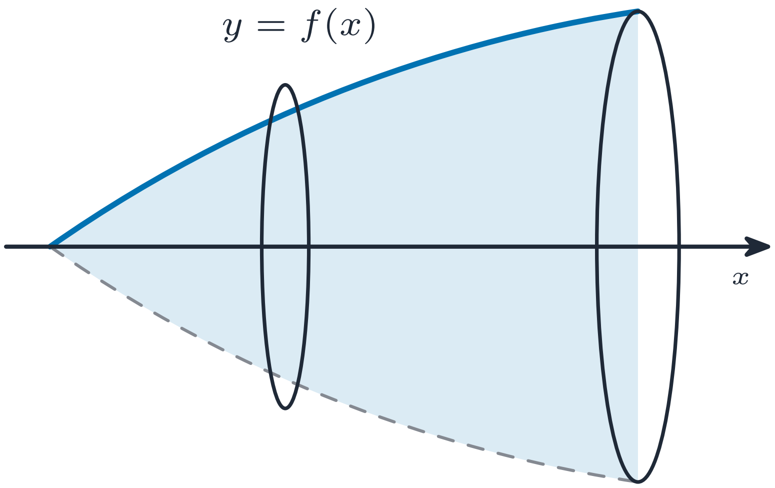
and $\displaystyle\int_0^1 x^{-1/2}\, dx = \left[2x^{1/2}\right]_0^1 = 2$, which is finite even though the integrand blows up at $x = 0$.

Volume of revolution

When a region is turned all the way around an axis it sweeps out a solid. The **volume of revolution** 旋转体体积 about the x-axis is

$$V = \pi \int_p^q y^2 \, dx,$$

and about the y-axis it is $V = \pi \displaystyle\int x^2 \, dy$. For example, the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$, turned about the x-axis, has volume $\pi \displaystyle\int_0^4 x \, dx = \pi \left[\frac{x^2}{2}\right]_0^4 = 8\pi$.



Rotating the region under a curve around the x-axis sweeps out a solid; each thin slice is a disc of area πy^2, so $V = \pi \int y^2 \, dx$

Exam tips

- Show **every line of algebra** —method marks are lost by jumping steps; use the **discriminant** $b^2 - 4ac$ to decide the number of real roots.
- Work in **radians** for circular measure (arc $= r\theta$, sector area $= \frac{1}{2}r^2\theta$) and for calculus of trig functions.
- For **differentiation**, set $\frac{dy}{dx} = 0$ for stationary points and use the second derivative to classify them.
- For **integration**, add $+c$ to an indefinite integral and use limits for area; an area below the axis gives a negative integral.

Pure Mathematics 2

A-Level Mathematics

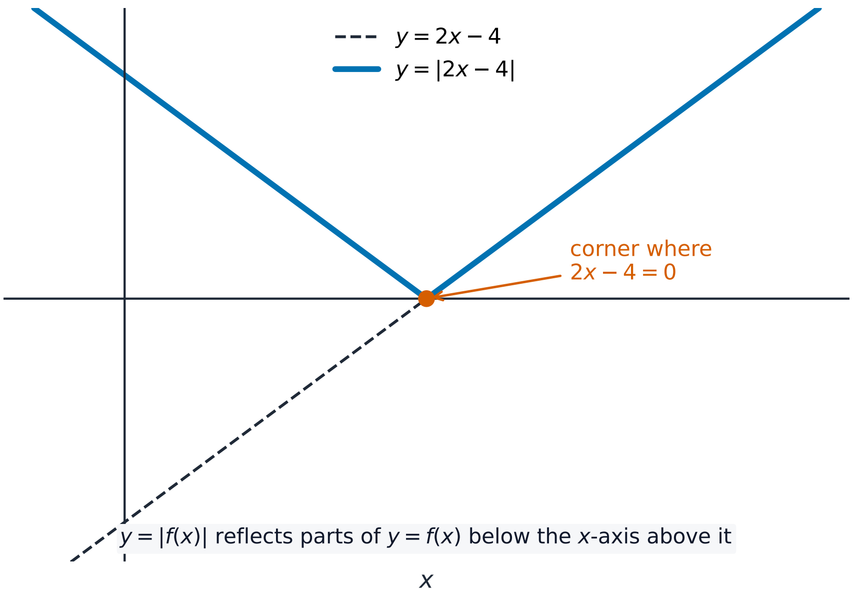
This handout covers Topic 2: **Pure Mathematics 纯数学 2**. It adds the modulus and polynomial algebra, **logarithms 对数** and the **exponential function 指数函数**, more trigonometry, and new ways to differentiate and integrate.

Algebra

The modulus

The **modulus 绝对值** $|x|$ is the size of a number with its sign removed, so $|x| \geqslant 0$ always. The graph of $y = |ax + b|$ is a "V" shape that bounces off the x-axis. Two useful rules for solving equations and inequalities are

$$|a| = |b| \;\Leftrightarrow\; a^2 = b^2, \qquad |x - a| < b \;\Leftrightarrow\; a - b < x < a + b.$$



Taking the modulus folds the part of the line below the axis upward into a V.

Worked example. Solve $|3x + 8| < 9$.

Using the second rule with the inequality written as $-9 < 3x + 8 < 9$:

$$-9 < 3x + 8 < 9 \;\Rightarrow\; -17 < 3x < 1 \;\Rightarrow\; -\tfrac{17}{3} < x < \tfrac{1}{3}.$$

Polynomial division and the factor and remainder theorems

A **polynomial** 多项式 is a sum of powers of x, such as $2x^4 + 3x^2 - 5$. Its **degree** 次数 is the highest power. When you divide one polynomial by another you get a **quotient** 商 and a **remainder** 余数.

- **Remainder theorem** 余数定理: the remainder when $p(x)$ is divided by $(x - a)$ is $p(a)$.
- **Factor theorem** 因式定理: $(x - a)$ is a factor of $p(x)$ exactly when $p(a) = 0$.

Worked example. The polynomial $p(x) = 2x^4 + kx^3 + kx^2 + 17x + 18$ has factor $(x + 2)$. Find k.

By the factor theorem $p(-2) = 0$:

$$2(16) + k(-8) + k(4) + 17(-2) + 18 = 0 \;\Rightarrow\; 16 - 4k = 0 \;\Rightarrow\; k = 4.$$

Logarithmic and exponential functions

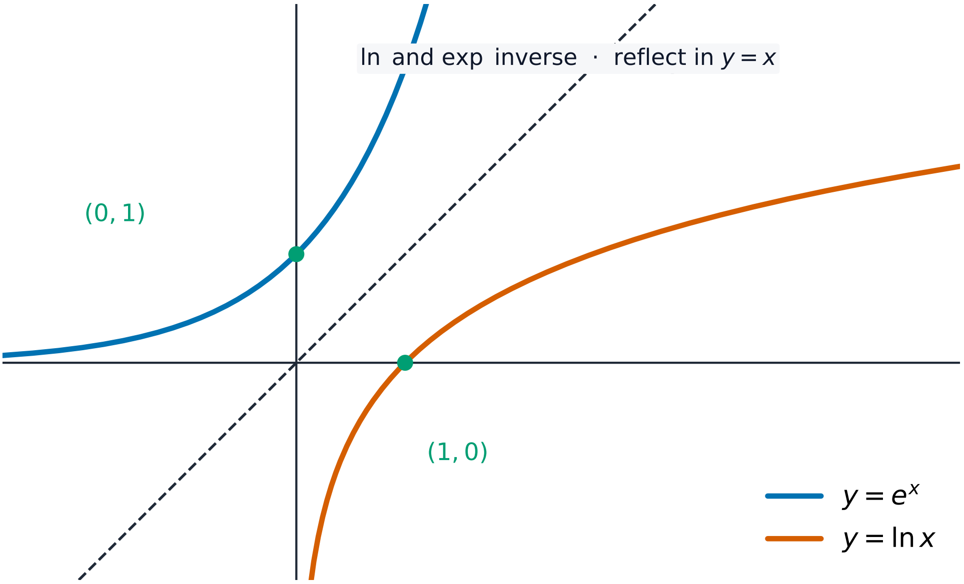
Earthquake strength is measured on the logarithmic Richter scale.

Populations can grow exponentially when resources are plentiful.

A **logarithm** answers the question ”what power?”. If $a^x = y$ then $x = \log_a y$. Logarithms and **indices** 指数 (powers) are reverse ideas. The **laws of logarithms** 对数定律 are

$$\log(mn) = \log m + \log n, \qquad \log\frac{m}{n} = \log m - \log n, \qquad \log(m^k) = k \log m.$$

The exponential function e^x and the **natural logarithm** 自然对数 $\ln x$ are inverse functions, so $\ln(e^x) = x$ and $e^{\ln x} = x$. When the unknown is in the power, take logs of both sides.



e^x and $\ln x$ undo each other, so each is the other reflected in $y = x$.

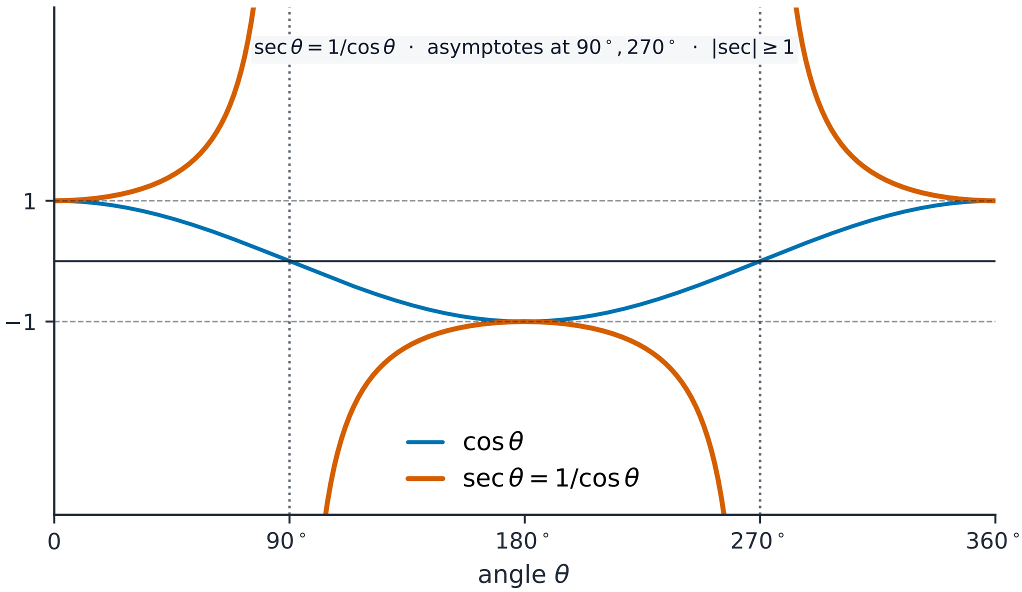
Worked example. Solve $4^x < 0.05$.

$$4^x < 0.05 \;\Rightarrow\; x \ln 4 < \ln 0.05 \;\Rightarrow\; x < \frac{\ln 0.05}{\ln 4} = -2.16 \text{ (3 s.f.)}.$$

Linear form 线性形式: a relationship like $y = Ax^n$ becomes a straight line if you take logs: $\ln y = \ln A + n \ln x$. Plotting $\ln y$ against $\ln x$ gives a line with gradient n and intercept $\ln A$, so you can find the unknown constants.

Trigonometry

There are three more functions, each the reciprocal of one you know: the **secant** 正割 $\sec\theta = \dfrac{1}{\cos\theta}$, the **cosecant** 余割 $\csc\theta = \dfrac{1}{\sin\theta}$, and the **cotangent** 余切 $\cot\theta = \dfrac{1}{\tan\theta}$.



$\sec\theta = 1/\cos\theta$ *rises to an asymptote wherever* $\cos\theta = 0$.

You must know these **trigonometric identities** 三角恒等式 and choose the right one for each problem:

$$\sec^2\theta \equiv 1 + \tan^2\theta, \qquad \csc^2\theta \equiv 1 + \cot^2\theta.$$

You also use the **compound angle** 复合角 formulae for $\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$, the **double angle** 二倍角 formulae

$$\sin 2A = 2\sin A\cos A, \quad \cos 2A = 2\cos^2 A - 1, \quad \tan 2A = \frac{2\tan A}{1 - \tan^2 A},$$

and the **R-formula** 辅助角公式 $a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)$, where $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \dfrac{b}{a}$.

Worked example. Solve $2\tan^2\theta + 3\sec\theta = 18$ for $-180° < \theta < 180°$.

Replace $\tan^2\theta$ with $\sec^2\theta - 1$ to get one function:

$2(\sec^2\theta - 1) + 3\sec\theta = 18 \;\Rightarrow\; 2\sec^2\theta + 3\sec\theta - 20 = 0 \;\Rightarrow\; (2\sec\theta - 5)(\sec\theta + 4) = 0.$

So $\sec\theta = \frac{5}{2}$ or $\sec\theta = -4$, giving $\cos\theta = \frac{2}{5}$ or $\cos\theta = -\frac{1}{4}$. The solutions are $\theta = \pm 66.4°$ and $\theta = \pm 104.5°$.

Differentiation

Learn these standard derivatives:

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\ln x = \frac{1}{x}, \quad \frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}\tan x = \sec^2 x.$$

For a product or a quotient of two functions, use:

- the **product rule** 乘积法则: $(uv)' = u'v + uv'$;
- the **quotient rule** 商法则: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

When a curve is given by **parametric equations** 参数方程 $x = x(t)$, $y = y(t)$, the gradient is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. When y is defined **implicitly** 隐式 (not made the subject), differentiate every term with respect to x, using the chain rule on the y terms, then solve for $\frac{dy}{dx}$.

Worked example. Given $y = 6x\cos(x^2+1)$, find $\frac{dy}{dx}$.

Use the product rule with $u = 6x$ and $v = \cos(x^2+1)$ (and the chain rule for v):

$$\frac{dy}{dx} = 6\cos(x^2+1) + 6x \cdot \big(-2x\sin(x^2+1)\big) = 6\cos(x^2+1) - 12x^2\sin(x^2+1).$$

Integration

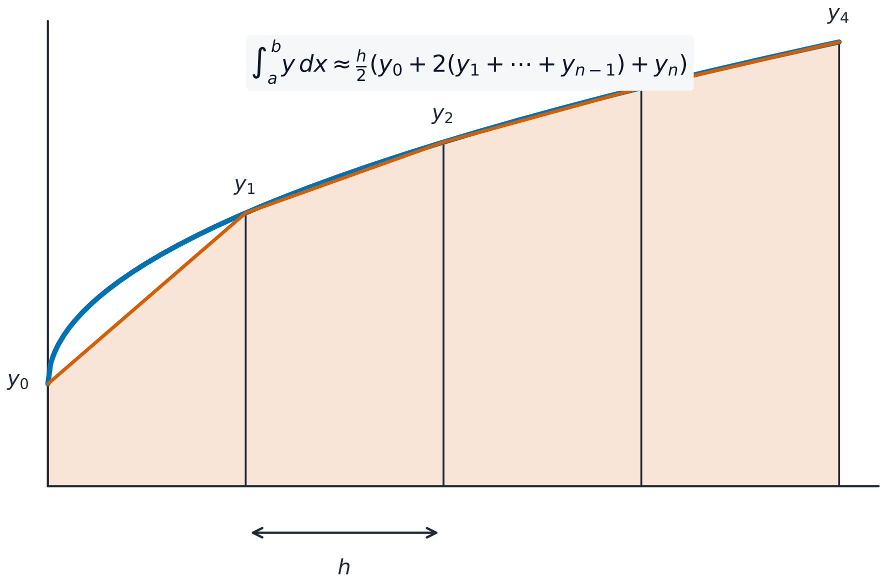
Integration is **reverse differentiation** —reverse each new derivative; harder integrals may need **integration by substitution** 换元积分 (developed in Pure 3). For a linear inside function $(ax+b)$:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C, \qquad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C,$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C, \qquad \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C, \qquad \int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C.$$

To integrate a power of sin or cos, first use an identity to remove the power. When you cannot integrate exactly, the **trapezium rule** 梯形法则 estimates a definite integral:

$$\int_a^b y\,dx \approx \tfrac{h}{2}\big[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\big].$$



Each strip of width h is a trapezium; their areas add up to estimate the integral.

Worked example. Find $\displaystyle\int 6\sin^2 x\,dx$.

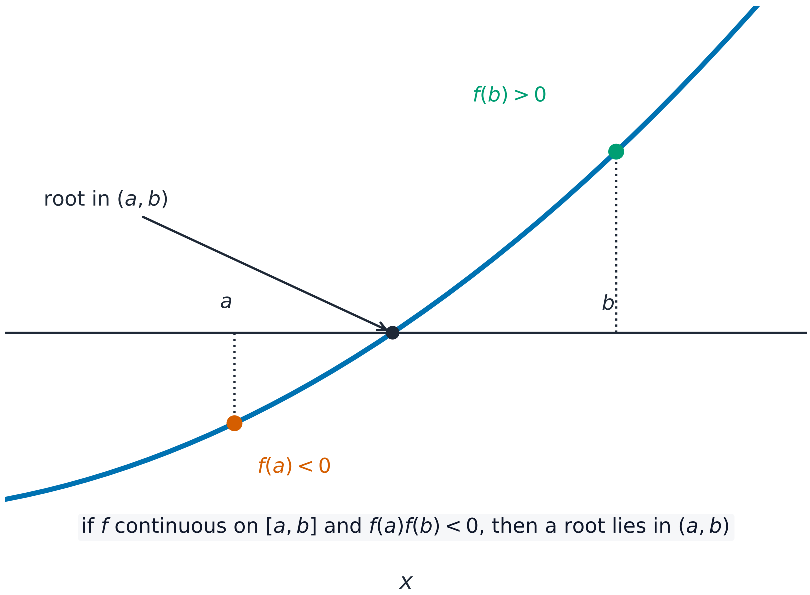
Use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$:

$$\int 6\sin^2 x\,dx = \int (3 - 3\cos 2x)\,dx = 3x - \tfrac{3}{2}\sin 2x + C.$$

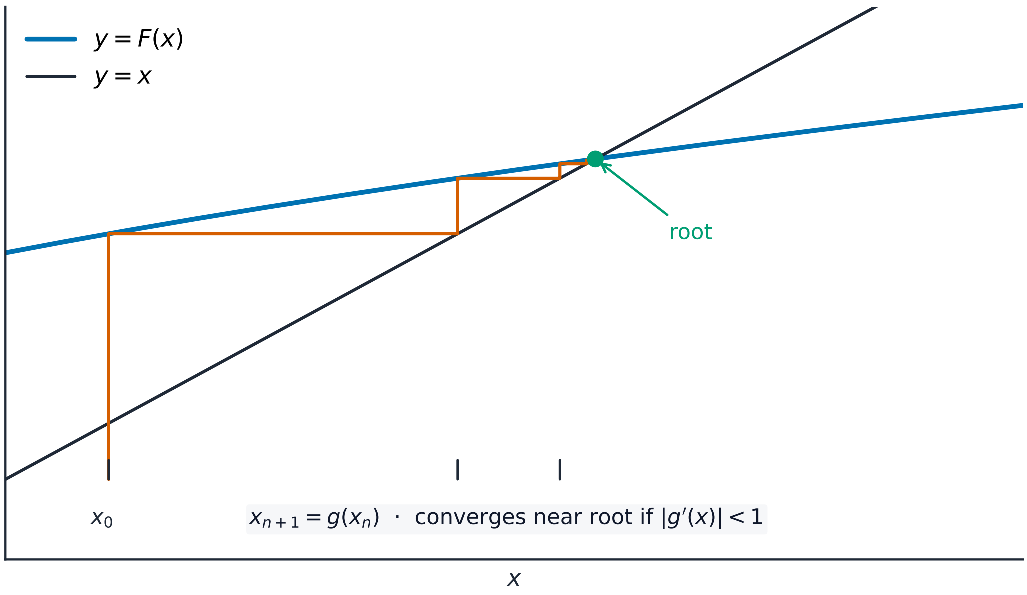
Numerical solution of equations

Many equations cannot be solved exactly. Two ideas help you find a **root** 根 (a solution).

- **Sign change** 变号: if $f(a)$ and $f(b)$ have opposite signs (and the graph has no break between them), a root lies between a and b.
- **Iteration** 迭代: rearrange the equation into the form $x = F(x)$, then use the **iterative formula** 迭代公式 $x_{n+1} = F(x_n)$. Start from a first guess x_0 and repeat. If the values are **convergent** they settle down and **converge** 收敛 to a root. Keep going until the answer is steady to the accuracy asked for.



$f(a)$ and $f(b)$ have opposite signs, so a root is trapped between a and b.



Each step goes up to $y = F(x)$ then across to $y = x$; the staircase closes in on the root.

Worked example. A root β of an equation satisfies $x = \sqrt[3]{-2x - 4.5}$, and $-1.4 < \beta < -1.0$. Use the iteration $x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ with $x_0 = -1.2$.

$$x_1 = \sqrt[3]{-2(-1.2) - 4.5} = \sqrt[3]{-2.1} = -1.281, \qquad x_2 = \sqrt[3]{-2(-1.281) - 4.5} = -1.247, \quad \ldots$$

The values settle near -1.26, so $\beta = -1.26$ (3 s.f.).

Exam tips

- Use the **laws of logarithms** to solve equations; remember ln and e^x are inverses.
- For **numerical methods**, show a **sign change** to locate a root, set out the iteration clearly, and give the answer to the stated accuracy.
- Learn the **chain, product and quotient rules** and identify which the function needs.
- Solve modulus equations $|f(x)| = g(x)$ by considering both the positive and negative cases, and sketch to check.

Pure Mathematics 3

A-Level Mathematics

This handout covers Topic 3: **Pure Mathematics** 纯数学 3. Subtopics 3.1–3.6 build on the algebra, logarithms, trigonometry, differentiation, integration and numerical methods of Pure Mathematics 2, so this handout explains what is **new** in Pure 3 and then covers the three big new areas: vectors, differential equations and complex numbers.

Algebra

Two new tools join the algebra from Pure 2.

Partial fractions

A single fraction with a factorised bottom can be split into a sum of simpler fractions. This is called writing it in **partial fractions** 部分分式, and it makes a **rational function** 有理函数 (a fraction of polynomials) easy to integrate or expand. Match the form to the bottom:

First check the top is **lower degree** than the bottom. If it is not ("top-heavy"), **divide** the polynomial first: dividing gives a **quotient** 商 plus a **remainder** 余数 over the original bottom, and you split only that remaining proper fraction.

$$\frac{x+4}{(x+1)(x-2)} \xrightarrow{\text{split}} \frac{2}{x-2} - \frac{1}{x+1}$$

One fraction with a factorised bottom splits into a sum of simpler fractions.

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}, \qquad \frac{1}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}.$$

Worked example. Express $\dfrac{x+4}{(x+1)(x-2)}$ in partial fractions.

Write $\dfrac{x+4}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$, so $x+4 = A(x-2) + B(x+1)$. Put $x = 2$: $6 = 3B$, so $B = 2$. Put $x = -1$: $3 = -3A$, so $A = -1$. Hence

$$\frac{x+4}{(x+1)(x-2)} = \frac{2}{x-2} - \frac{1}{x+1}.$$

The binomial expansion for a rational power

The **binomial expansion** 二项展开式 also works when the power is a fraction or is negative, as long as $|x| < 1$:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

For example $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$ for $|x| < 1$.

Logarithms, trigonometry and numerical methods (from Pure 2)

Subtopics 3.2, 3.3 and 3.6 are the same skills you met in Pure 2: the laws of logarithms with e^x and $\ln x$; the identities $\sec^2\theta \equiv 1+\tan^2\theta$ and $\csc^2\theta \equiv 1+\cot^2\theta$, the compound- and double-angle formulae, and the R-form of $a\sin\theta + b\cos\theta$; and solving an equation numerically by a **sign change** 变号 and an **iterative formula** 迭代公式 $x_{n+1} = F(x_n)$. Use them exactly as before.

The three new trig functions are the **reciprocals** 倒数 of the familiar ones: the **secant** 正割 $\sec\theta = \dfrac{1}{\cos\theta}$, the **cosecant** 余割 $\csc\theta = \dfrac{1}{\sin\theta}$, and the **cotangent** 余切 $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$. (Memory aid: match the *third* letter —se**c** goes with **c**osine.) The two Pythagorean identities above come straight from dividing $\sin^2\theta + \cos^2\theta \equiv 1$ by $\cos^2\theta$ or $\sin^2\theta$. For a numerical method, an iteration $x_{n+1} = F(x_n)$ **converges** 收敛 to the root when successive values get closer together.

Differentiation

The methods are those of Pure 2 (the product, quotient and chain rules, with parametric and implicit curves). One derivative is added —the **inverse tangent** 反正切:

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}.$$

Worked example. Differentiate $y = \tan^{-1}(3x)$.

Use the chain rule with the result above: $\dfrac{dy}{dx} = \dfrac{1}{1+(3x)^2} \times 3 = \dfrac{3}{1+9x^2}$.

Integration

Pure 3 adds several powerful methods.

- A new standard integral: $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$.
- **Partial fractions**: split a rational function first, then integrate each piece as a logarithm.
- The pattern $\dfrac{k\,f'(x)}{f(x)}$: this integrates to $k\ln|f(x)|+C$. For example $\displaystyle\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1)+C$.
- **Integration by parts** 分部积分, used for a product: $\displaystyle\int u\,\frac{dv}{dx}\,dx = uv - \int v\,\frac{du}{dx}\,dx$.
- **Integration by substitution** 换元积分: a given change of variable turns a hard integral into an easy one.

Worked example. Find $\displaystyle\int x\cos x\,dx$.

Use integration by parts with $u = x$ and $\dfrac{dv}{dx} = \cos x$, so $\dfrac{du}{dx} = 1$ and $v = \sin x$:

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Vectors

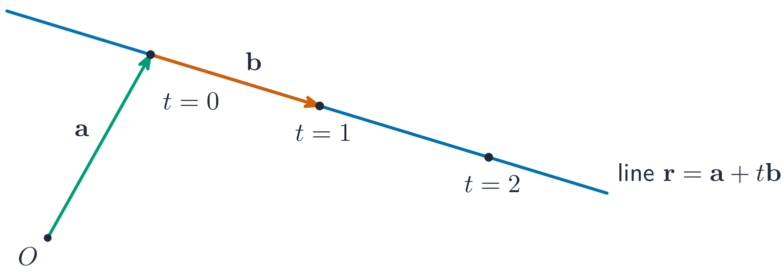
Forces like wind and water are vectors —they have both size and direction.

A **vector** 向量 has both size and direction. Write it as a column, or as $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, or as $\overrightarrow{AB}$.

- The **magnitude** 模长 (length) of $\mathbf{v}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ is $|\mathbf{v}|=\sqrt{x^2+y^2+z^2}$.
- A **unit vector** 单位向量 has magnitude 1; divide a vector by its magnitude to make one.
- A **position vector** 位置向量 gives a point's place from the origin; a **displacement vector** 位移向量 $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$ goes from one point to another. Multiplying by a **scalar** 标量 (a plain number) stretches a vector.

Lines and the scalar product

A straight line through point $\mathbf{a}$ in direction $\mathbf{b}$ has vector equation $\mathbf{r}=\mathbf{a}+t\mathbf{b}$. Two lines may be **parallel** 平行, may **intersect** 相交 at a point, or may be **skew lines** 异面直线 (not parallel and never meeting).

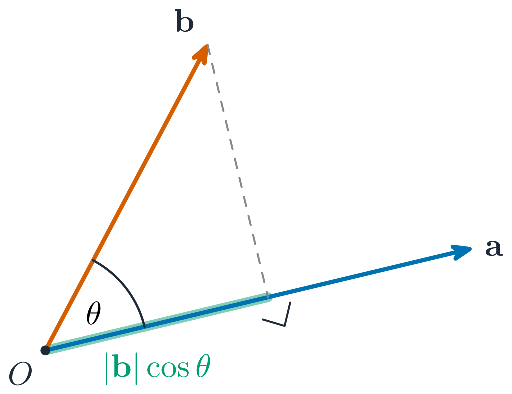


Start at the point $\mathbf{a}$*, then add* t *copies of the direction* $\mathbf{b}$ *to reach any point on the line.*

The **scalar product** 数量积 (dot product) of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3=|\mathbf{a}|\,|\mathbf{b}|\cos\theta,$$

where θ is the angle between them. So $\mathbf{a}\cdot\mathbf{b}=0$ means the vectors are perpendicular.



The scalar product picks out $|\mathbf{b}|\cos\theta$, how far $\mathbf{b}$ reaches along $\mathbf{a}$.

Worked example. Find the angle between $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

$$\mathbf{a}\cdot\mathbf{b} = (1)(2) + (2)(2) + (2)(1) = 8, \qquad |\mathbf{a}| = |\mathbf{b}| = 3.$$

So $\cos\theta = \dfrac{8}{3\times 3} = \dfrac{8}{9}$, giving $\theta = 27.3°$.

Differential equations

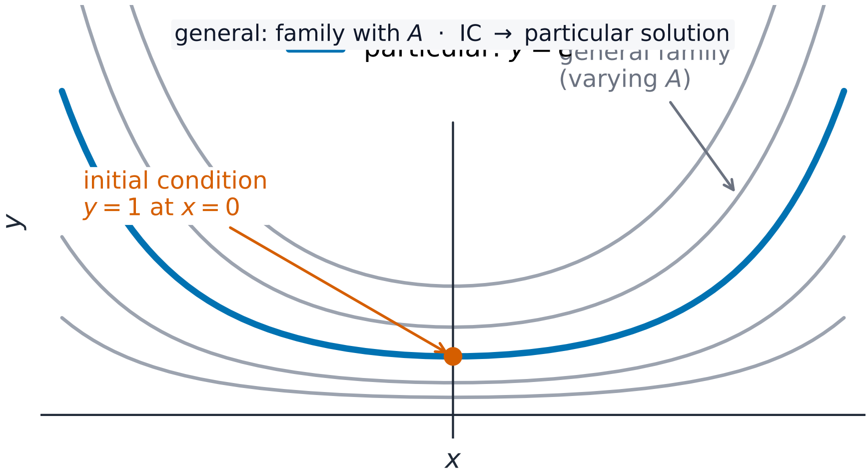
A **differential equation** 微分方程 links a quantity to its **rate of change**. To solve a first-order equation whose variables are **separable** 可分离变量, put all the y terms on one side and all the x terms on the other, then integrate both sides. This gives the **general solution** 通解, which contains a constant. An **initial condition** 初始条件 (a known value) fixes the constant and gives the **particular solution** 特解.

Worked example. Solve $\dfrac{dy}{dx} = xy$, given that $y = 1$ when $x = 0$.

Separate the variables and integrate:

$$\int \frac{1}{y}\,dy = \int x\,dx \;\Rightarrow\; \ln y = \tfrac{1}{2}x^2 + c \;\Rightarrow\; y = Ae^{x^2/2}.$$

Using $y = 1$ at $x = 0$ gives $A = 1$, so $y = e^{x^2/2}$.



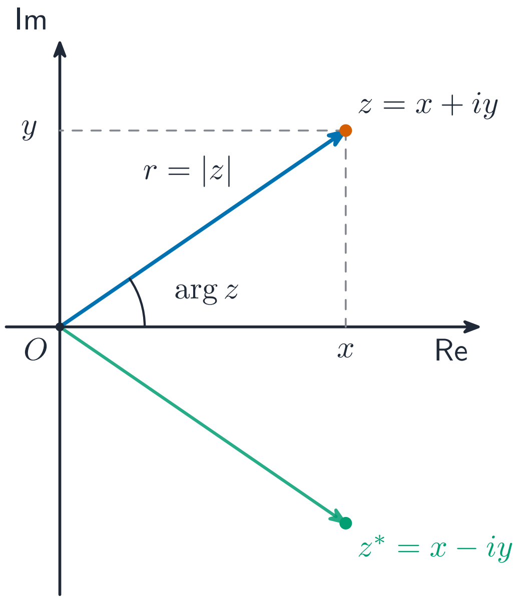
The constant A gives a whole family of curves; the condition $y = 1$ at $x = 0$ selects $y = e^{x^2/2}$.

Complex numbers

Self-similar patterns like Romanesco arise from iterating functions in the complex plane.

A **complex number** 复数 has the form $z = x + iy$, where $i^2 = -1$. Here x is the **real part** 实部 and y is the **imaginary part** 虚部. This $x + iy$ is the **Cartesian form** 直角坐标形式. Two complex numbers are equal only when their real parts match and their imaginary parts match.

- The **conjugate** 共轭 of $z = x + iy$ is $z^* = x - iy$. For a polynomial with real coefficients, any non-real roots come in conjugate pairs.
- The **modulus** 模 is $|z| = \sqrt{x^2 + y^2}$ (its distance from the origin) and the **argument** 辐角 is the angle the point makes, measured from the positive real axis.
- You can plot z as a point on an **Argand diagram** 阿干图 (the complex plane).
- The **polar form** 极坐标形式 is $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where $r = |z|$ and θ is the argument. Multiplying multiplies the moduli and adds the arguments.
- The **loci** of points satisfying a condition on z are drawn on the Argand diagram —e.g. $|z - a| = r$ is a circle, $|z - a| = |z - b|$ a perpendicular bisector, and $\arg(z - a) = \theta$ a half-line. The **square roots** of a complex number come from solving $w^2 = z$.



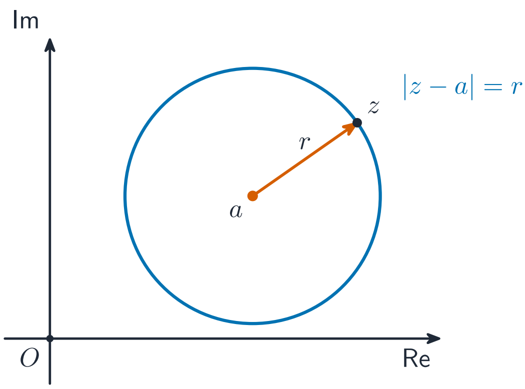
On the Argand diagram $|z|$ is the distance from O, $\arg z$ the angle, and z^* the reflection in the real axis.

To divide, multiply top and bottom by the conjugate of the bottom.

Worked example. Write $\dfrac{3+i}{1-i}$ in the form $x + iy$.

$$\frac{3+i}{1-i} = \frac{(3+i)(1+i)}{(1-i)(1+i)} = \frac{3+3i+i+i^2}{1+1} = \frac{2+4i}{2} = 1+2i.$$

An equation or inequality in z describes a **locus** 轨迹 (a path or region) on the Argand diagram. For example $|z-a| = r$ is a circle of radius r centred at a.



$|z-a| = r$ is the set of points a fixed distance r from a —a circle.

Exam tips

- Split a rational function into **partial fractions** before integrating or expanding.
- Choose the right **integration technique** (substitution, by parts, or partial fractions) from the form of the integrand.
- Use the scalar (dot) product for the **angle between vectors** and to test for perpendicularity; write a line as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.
- Give **complex numbers** in the form asked for (Cartesian or modulus-argument) and show them on an Argand diagram.

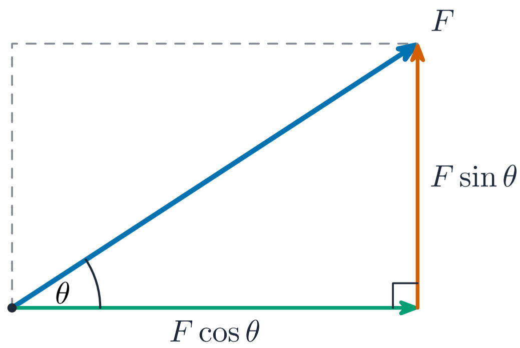
Mechanics

A-Level Mathematics

This handout covers Topic 4: **Mechanics 力学**. It studies how forces make objects move. Throughout this topic, take the acceleration of free fall as $g = 10$ m s^{-2}.

Forces and equilibrium

A **force 力** is a push or a pull. It is a vector, so it has size and direction. Because it is a vector, you can split a force into **components 分量** (usually horizontal and vertical), and you can add several forces into one **resultant 合力**.



A force at angle θ has a horizontal part $F\cos\theta$ and a vertical part $F\sin\theta$.

A particle is in **equilibrium 平衡** when the forces are balanced: the vector sum of the forces is zero. In practice this means the components in any direction add to zero.

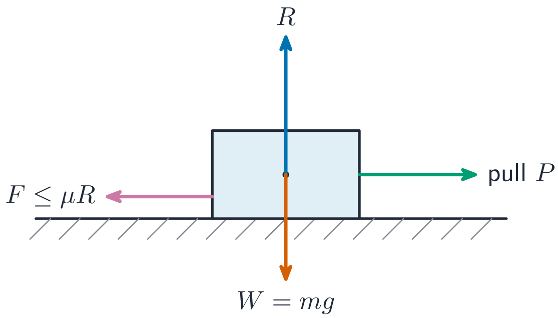
Every mechanics problem is a picture like this. Three forces act on the climber — weight straight down, tension along the rope, and a push from the rock —and because they balance, the climber hangs still in equilibrium. Resolving each into horizontal and vertical parts turns the picture into equations

Friction

When two surfaces touch, the **contact force 接触力** between them has two parts: the **normal reaction 法向反作用力** R, at right angles to the surface, and the **friction 摩擦力** F, along the surface, which opposes sliding. A "smooth" surface is a model with no friction.

Friction can only grow up to a maximum. At that maximum the body is in **limiting equilibrium 极限平衡**, about to slip, and the friction is **limiting friction 最大静摩擦力**. The maximum is set by the **coefficient of friction 摩擦系数** μ:

$$F \leqslant \mu R, \qquad \text{with } F = \mu R \text{ at the point of slipping.}$$



On a rough surface the contact force splits into the normal reaction R and friction F (at most μR).

By **Newton's third law 牛顿第三定律**, the two surfaces push on each other with equal and opposite forces.

Worked example. A block of weight 20 N rests on a rough horizontal table with coefficient of friction $\mu = 0.4$. Find the largest horizontal force that can be applied before the block slides.

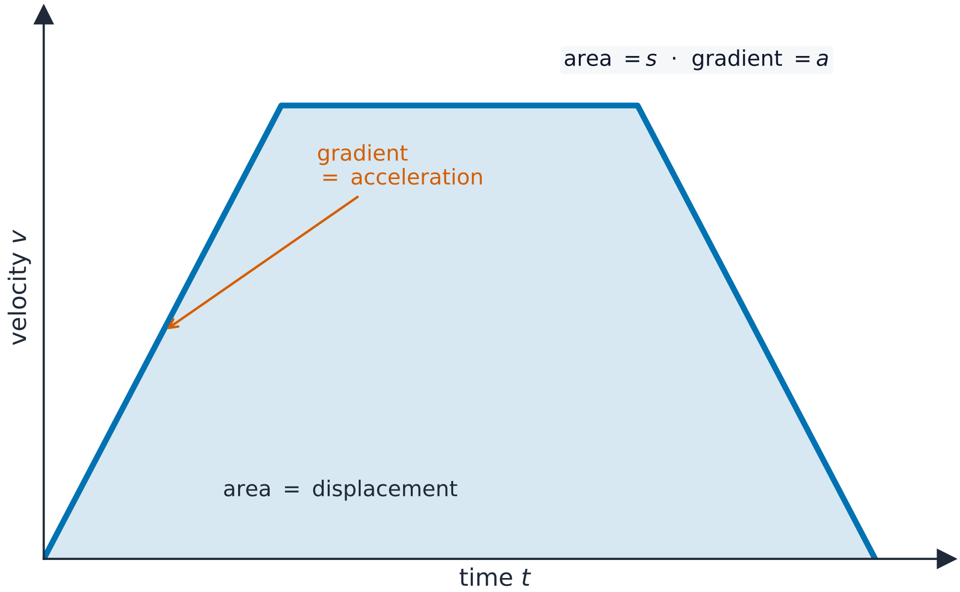
The table's normal reaction balances the weight, so $R = 20$ N. The block is on the point of slipping when friction reaches its maximum $F = \mu R = 0.4 \times 20 = 8$ N. In equilibrium the applied force equals the friction, so the largest force it can resist is 8 N.

Kinematics of motion in a straight line

Distance 距离 and **speed 速率** are scalars (size only). **Displacement 位移**, **velocity 速度** and **acceleration 加速度** are vectors (size and direction).

On a **velocity-time graph 速度时间图**, the area under the graph is the displacement and the gradient is the acceleration (a negative acceleration is a **deceleration 减速**

度). On a displacement-time graph, the gradient is the velocity. More generally, differentiate with respect to time to go from displacement to velocity to acceleration, and integrate to go back.



The shaded area gives the distance travelled; the slope of the line gives the acceleration.

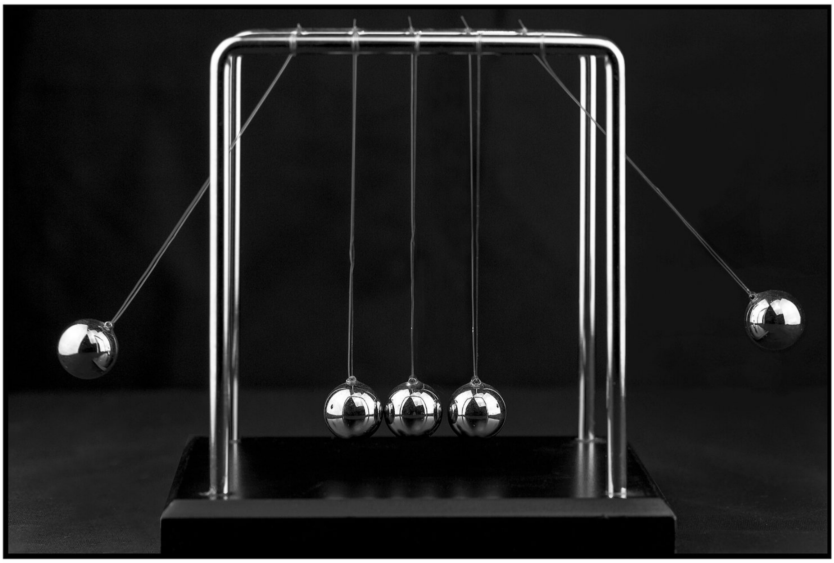
For motion with **constant acceleration** 匀加速, use these formulae (the "suvat" equations):

$$v = u + at, \qquad s = ut + \tfrac{1}{2}at^2, \qquad v^2 = u^2 + 2as, \qquad s = \tfrac{1}{2}(u+v)t.$$

Worked example. A car starts from rest and accelerates at 2.5 m s^{-2} for 4 s. Find its speed and the distance travelled.

$$v = 0 + 2.5 \times 4 = 10 \text{ m s}^{-1}, \qquad s = 0 + \tfrac{1}{2}(2.5)(4^2) = 20 \text{ m}.$$

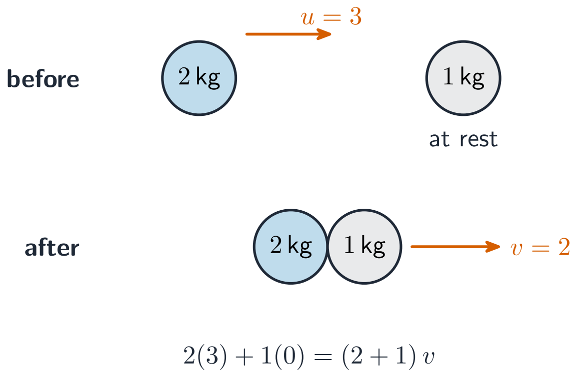
Momentum



A Newton's cradle demonstrates conservation of momentum in collisions.

The **linear momentum** 动量 of a body is mass $\times$ velocity. It is a vector. In a direct collision of two bodies, the total momentum is unchanged. This is the **conservation of linear momentum** 动量守恒:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2.$$



The total momentum is the same before and after the collision.

Worked example. A body of mass 2 kg moving at 3 m s^{-1} hits a stationary body of mass 1 kg, and they stick together. Find their common speed afterwards.

$$2(3) + 1(0) = (2+1)v \;\Rightarrow\; v = \frac{6}{3} = 2 \text{ m s}^{-1}.$$

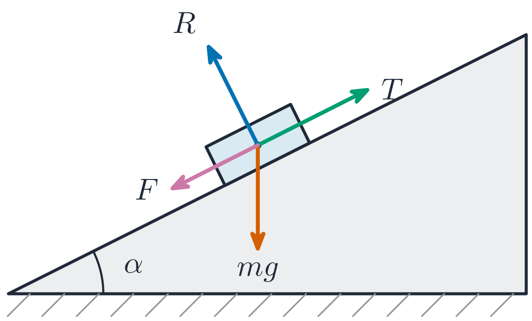
Newton's laws of motion

Newton's laws of motion 牛顿运动定律 connect force and acceleration. The key one is: resultant force = **mass** 质量 × acceleration,

$$F = ma.$$

The **weight** 重力 of a body is the force of gravity on it: $W = mg$. Forces in a problem may include weight, friction, **tension** 张力 in a string, the **thrust** 推力 (push) in a rod, and **air resistance** 空气阻力.

For motion on an **inclined plane** 斜面, split each force into a part along the slope and a part at right angles to it, then apply $F = ma$ along the slope. For **connected particles** 连接质点 (joined by a string), apply $F = ma$ to each body, or to the whole system.



On a slope, resolve the forces along the slope and at right angles to it.

Worked example. A block of mass 12 kg is pulled up a rough plane by a rope parallel to the slope. The plane is at 20° to the horizontal, the coefficient of friction is 0.4, and the acceleration is 2 m s^{-2}. Find the tension in the rope.

The normal reaction is $R = mg\cos 20° = 120\cos 20° = 112.8$ N, so the friction is $F = \mu R = 0.4 \times 112.8 = 45.1$ N. Along the slope, $T - mg\sin 20° - F = ma$:

$$T = ma + mg\sin 20° + F = 12(2) + 120\sin 20° + 45.1 = 24 + 41.0 + 45.1 = 110 \text{ N (3 s.f.)}.$$

Energy, work and power

A roller coaster trades potential energy for kinetic energy as it rises and falls.

The **work done** 功 by a constant force is the **scalar product** of force and displacement —the force times the distance moved in the direction of the force: $W = Fd\cos\theta$, where θ is the angle between the force and the motion. Work is measured in joules (J).

Energy comes in forms you can calculate:

- **Kinetic energy** 动能 (energy of movement): $\mathrm{KE} = \frac{1}{2}mv^2$.
- **Gravitational potential energy** 重力势能 (energy of height): $\mathrm{PE} = mgh$.

The work done by the outside forces equals the change in the total energy. When no friction acts, the total energy stays the same —the **conservation of energy** 能量守恒.

Power 功率 is the rate of doing work. For a force pulling in the direction of motion, $P = Fv$ (power = force × velocity). Power is measured in watts (W).

Worked example. A car engine works at 12 kW while the car moves at 20 m s^{-1} on a level road. Find the driving force.

$$P = Fv \;\Rightarrow\; F = \frac{P}{v} = \frac{12000}{20} = 600 \text{ N}.$$

Exam tips

- Draw a clear **force diagram** and resolve into perpendicular components; for equilibrium, each direction sums to zero.
- Use **SUVAT** only for constant acceleration and keep a consistent positive direction.
- Apply $F = ma$ along the direction of motion, including friction ($F = \mu R$) on a rough surface.
- State your **assumptions** (light inextensible string, smooth pulley, particle) —they are often worth a mark.

Probability & Statistics 1

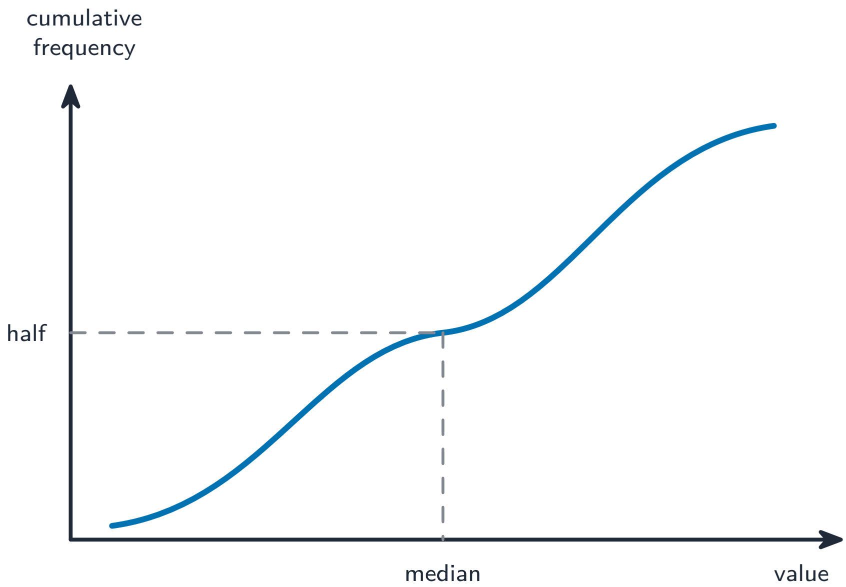
A-Level Mathematics

This handout covers Topic 5: **Probability & Statistics 概率统计** 1. It is about describing data, counting choices, and working out the chance of events.

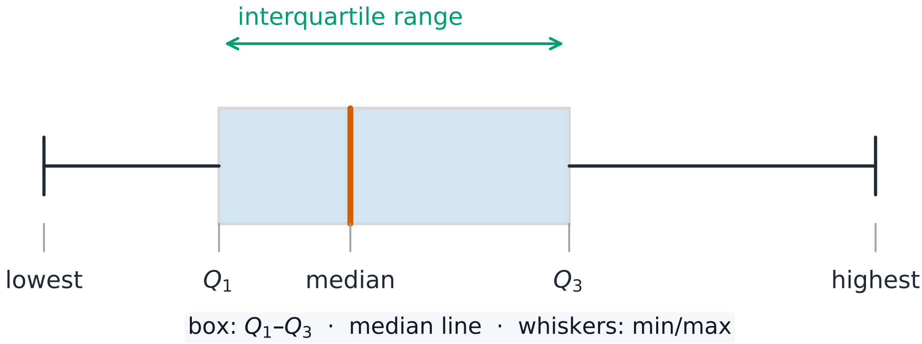
Representation of data

Choose a diagram that suits the data. You should be able to draw and read:

- a **stem-and-leaf diagram** 茎叶图 (keeps the original values and shows the shape); two data sets are compared with a **back-to-back** 背靠背 version –a shared central stem, one set's leaves increasing to the left and the other's to the right, so you can compare their medians and spreads at a glance;
- a **box-and-whisker plot** 箱线图 (shows the lowest value, the three quartiles, and the highest value);
- a **histogram** 直方图 (for grouped data, where the **area** of each bar shows the frequency);
- a **cumulative frequency** 累积频数 graph (running totals, used to estimate the median and quartiles).



A cumulative frequency curve is S-shaped; read the median across from half the total frequency, and the quartiles from one and three quarters



The box spans the quartiles Q_1 to Q_3; the whiskers reach the lowest and highest values.

Averages and spread

A **measure of central tendency 集中趋势** is a single "middle" value:

- the **mean 平均数** $\bar{x} = \dfrac{\sum x}{n}$ (the average);
- the **median 中位数** (the middle value when the data is in order);
- the **mode 众数** (the most common value).

A measure of **variation 离散程度** shows how spread out the data is:

- the **range (of data) 极差** (highest − lowest);
- the **interquartile range 四分位距** (upper quartile − lower quartile);
- the **standard deviation 标准差** $\sigma = \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2}$.

You often work from the totals $\sum x$ and $\sum x^2$. The square of the standard deviation is the **variance 方差**.

Worked example. For 10 values, $\sum x = 50$ and $\sum x^2 = 300$. Find the mean and standard deviation.

$$\bar{x} = \frac{50}{10} = 5, \qquad \sigma = \sqrt{\frac{300}{10} - 5^2} = \sqrt{30 - 25} = \sqrt{5} = 2.24.$$

Coding 编码 makes big numbers easier. Replace each value by $t = x - a$ for a convenient **assumed mean 假定平均数** a. Then $\bar{x} = a + \bar{t}$, while the standard deviation is **unchanged** (shifting every value does not spread the data). So from the coded totals $\sum(x-a)$ and $\sum(x-a)^2$ you get $\bar{x}$ and σ directly, and two data sets can be compared through their coded totals.

Permutations and combinations

A **permutation** 排列 is an arrangement where order matters; a **combination** 组合 is a selection where order does not matter. The numbers are

$$^{n}P_r = \frac{n!}{(n-r)!}, \qquad {}^{n}C_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

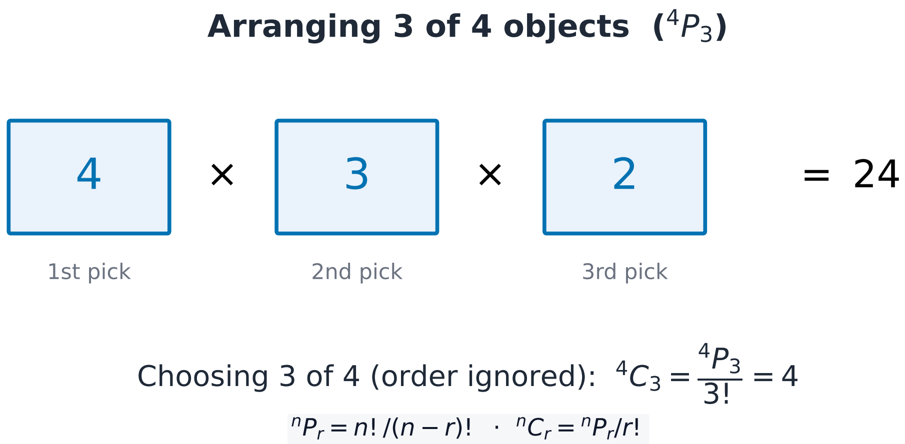
To arrange objects in a line when some are repeated, divide by the factorial of each repeat count.

Order matters for a permutation, but not for a combination

Worked example. How many different arrangements are there of the letters of the word NEEDLESS?

There are 8 letters, with E repeated 3 times and S repeated 2 times:

$$\frac{8!}{3!\,2!} = \frac{40320}{6\times 2} = 3360.$$



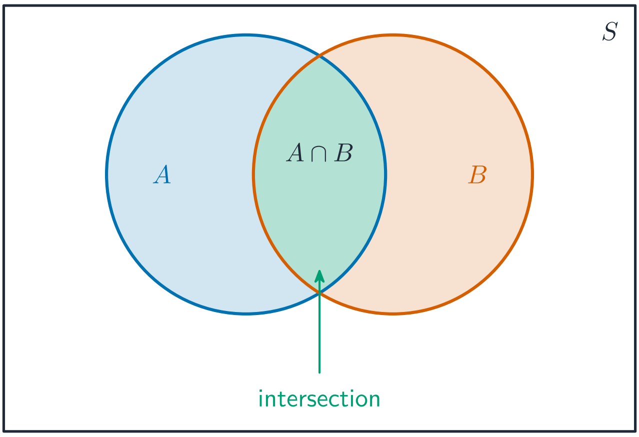
$nPr = n!/(n{-}r)!;\ nCr = nPr/r!$

Probability

Dice: a familiar starting point for probability.

Find a probability by counting equally likely outcomes, or by using permutations and combinations. Combine probabilities with these rules:

- **addition** for "or": $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;
- **multiplication** for "and" when events are independent: $P(A \cap B) = P(A)\,P(B)$.



The overlap of the two circles is $A \cap B$; the addition rule subtracts it once so it is not counted twice.

Two events are **mutually exclusive events** 互斥事件 if they cannot both happen, and **independent events** 独立事件 if one happening does not change the chance of the other. To test independence, check whether $P(A \cap B) = P(A) \times P(B)$. A **conditional probability** 条件概率 is the chance of A given that B has happened: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$.

Worked example. Events have $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cap B) = 0.2$. Are A and B independent?

Test: $P(A) \times P(B) = 0.5 \times 0.4 = 0.2 = P(A \cap B)$. The values are equal, so A and B are independent.

Discrete random variables

A **discrete random variable** 离散型随机变量 X takes separate values, each with a probability. List them in a **probability distribution table** 概率分布表; the probabilities must add to 1. Then the **expectation** 期望 (mean) and **variance** are

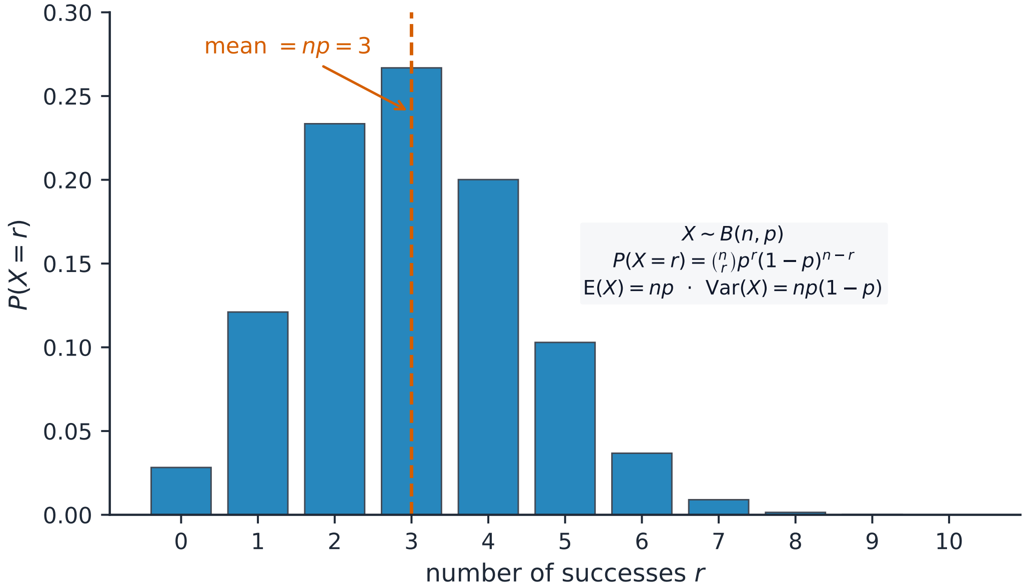
$$E(X) = \sum x\, P(X = x), \qquad \mathrm{Var}(X) = \sum x^2\, P(X = x) - \big(E(X)\big)^2.$$

Two special models:

- the **binomial distribution** 二项分布 $X \sim B(n, p)$, for the number of successes in n independent trials: $P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$, with $E(X) = np$ and $\mathrm{Var}(X) = np(1-p)$;
- the **geometric distribution** 几何分布, for the trial on which the first success happens: $P(X = r) = (1-p)^{r-1} p$, with $E(X) = \dfrac{1}{p}$.

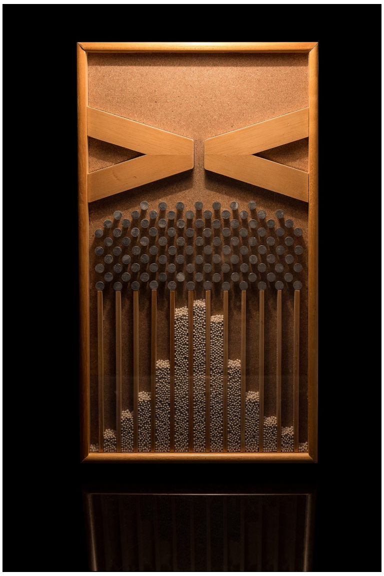
Worked example. $X \sim B(10, 0.3)$. Find $P(X = 2)$ and $E(X)$.

$$P(X = 2) = \binom{10}{2} (0.3)^2 (0.7)^8 = 45 \times 0.09 \times 0.05765 = 0.233, \qquad E(X) = 10 \times 0.3 = 3.$$



The distribution $B(10, 0.3)$: each bar is $P(X = r)$, clustered around the mean $np = 3$.

The normal distribution



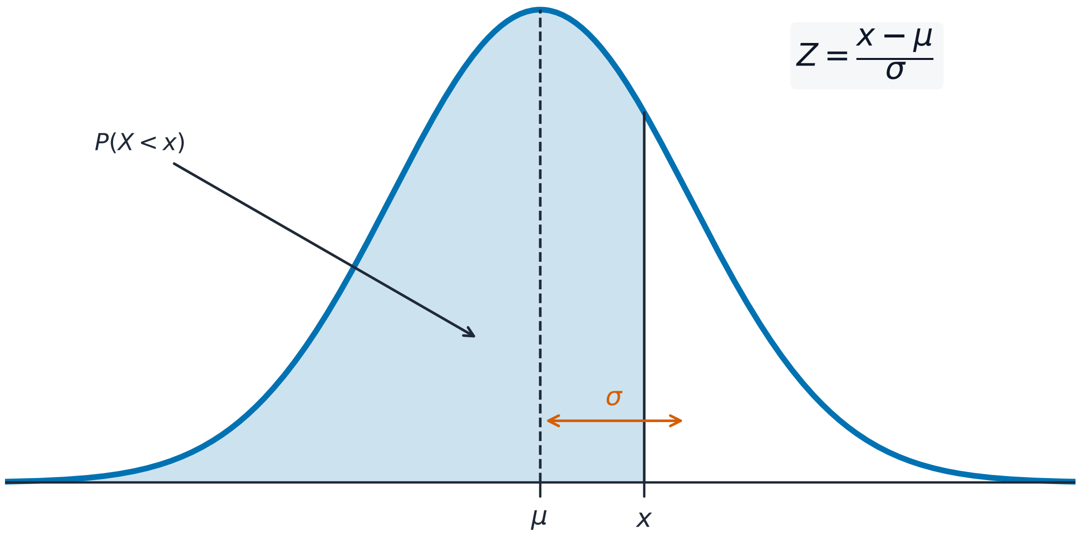
A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.

The **normal distribution** 正态分布 models a **continuous random variable** 连续型随机变量 with a symmetric bell shape. Write $X \sim N(\mu, \sigma^2)$, where μ is the mean and σ is the standard deviation. To use the tables, by **standardisation** 标准

化 convert to the variable $Z \sim N(0,1)$:

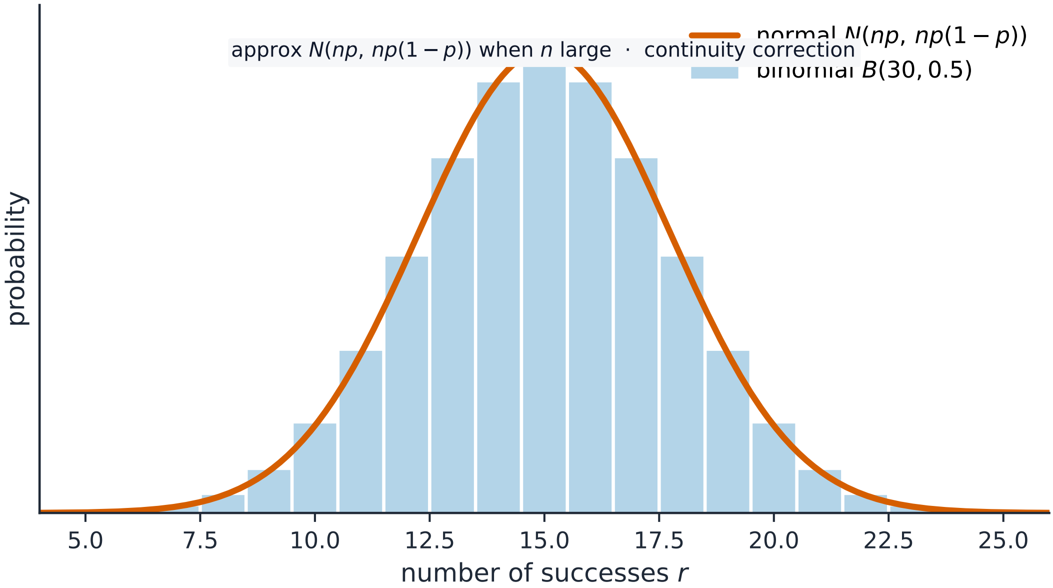
$$Z = \frac{X-\mu}{\sigma}.$$

Then $P(X<x) = P\left(Z < \frac{x-\mu}{\sigma}\right)$, which you read from the normal table Φ.



A normal probability is an area under the bell curve; standardizing rescales it to $Z \sim N(0,1)$.

The normal distribution is also a good **approximation** 近似 to the binomial when n is large. Because you replace a discrete variable by a continuous one, apply a **continuity correction** 连续性校正 (adjust by 0.5).



When n is large the binomial bars follow a normal curve of the same mean and variance.

Worked example. Bags of rice have mass $X \sim N(\mu, 0.14^2)$. Given that $P(X < 1.48) = 0.22$, find μ.

From the table, $P(Z < z) = 0.22$ gives $z = -0.772$. So

$$\frac{1.48 - \mu}{0.14} = -0.772 \;\Rightarrow\; \mu = 1.48 + 0.772 \times 0.14 = 1.59 \text{ kg (3 s.f.)}.$$

Exam tips

- Decide whether order matters: **permutations** (^nP_r) when it does, **combinations** (^nC_r) when it does not.
- For a **discrete random variable**, check the probabilities sum to 1 and use $E(X) = \sum x\,P(X = x)$.
- For the **normal distribution**, standardise with $z = (x-\mu)/\sigma$, **sketch and shade**, then read the table.
- Apply a **continuity correction** when approximating a discrete variable by the normal.

Probability & Statistics 2

A-Level Mathematics

This handout covers Topic 6: **Probability & Statistics** 概率统计 2. It adds the Poisson model, combining random variables, continuous distributions, and the ideas of estimation and testing.

The Poisson distribution

People arriving at random in a queue follow a Poisson distribution.

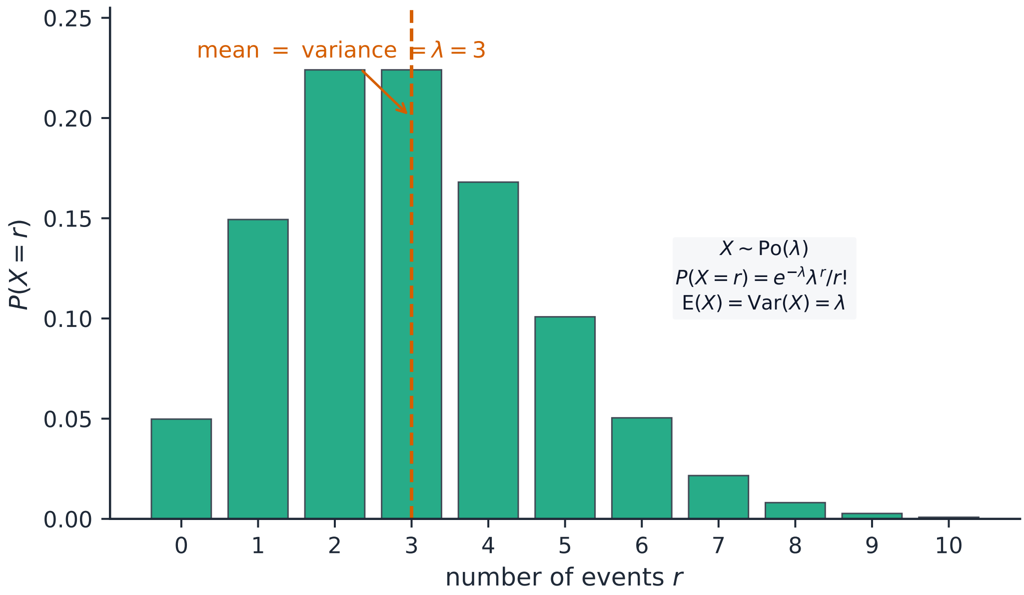
The **Poisson distribution** 泊松分布 $X \sim \mathrm{Po}(\lambda)$ models the number of random events in a fixed interval, when events happen at a steady average rate λ:

$$P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!}.$$

For a Poisson variable the mean and the variance are both equal to λ. The Poisson distribution is a good **approximation** 近似 to the **binomial distribution** when n is large and p is small. The normal distribution (with continuity correction) approximates the Poisson when λ is large.

Worked example. $X \sim \mathrm{Po}(3)$. Find $P(X = 2)$.

$$P(X = 2) = e^{-3}\frac{3^2}{2!} = e^{-3} \times 4.5 = 0.224.$$



The Poisson distribution Po(3)*: for a Poisson variable the mean and variance both equal* λ*.*

Linear combinations of random variables

When you change a variable by a linear rule, the **expectation** 期望 (mean) and **variance** 方差 follow these rules:

$$E(aX+b) = aE(X)+b, \qquad \mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X).$$

For two **independent** variables X and Y:

$$E(aX+bY) = aE(X)+bE(Y), \qquad \mathrm{Var}(aX+bY) = a^2\,\mathrm{Var}(X)+b^2\,\mathrm{Var}(Y).$$

Two useful facts: if X has a **normal distribution** 正态分布 then so does $aX+b$; and the sum of independent Poisson variables is again Poisson.

Worked example. X has mean 5 and variance 4. Find $E(3X-1)$ and $\mathrm{Var}(3X-1)$.

$$E(3X-1) = 3(5)-1 = 14, \qquad \mathrm{Var}(3X-1) = 3^2\times 4 = 36.$$

Continuous random variables

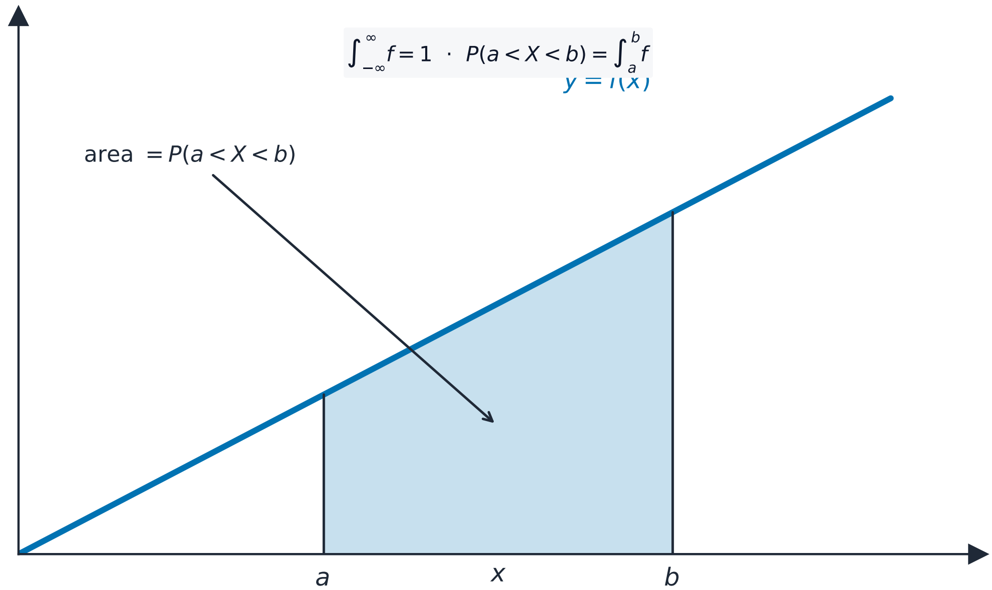
A **continuous random variable** 连续型随机变量 can take any value in a range. Its probabilities come from a **probability density function** 概率密度函数 $f(x)$, with two key properties:

$$f(x) \geqslant 0, \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

A probability is the area under f, and the mean is found by integration:

$$P(a < X < b) = \int_a^b f(x)\,dx, \qquad E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx.$$

The **cumulative distribution function** 累积分布函数 is $F(x) = P(X \leqslant x) = \int_{-\infty}^{x} f(t)\,dt$; the **median** 中位数 solves $F(m) = 0.5$, and other **percentiles** 百分位数 solve $F(x) = p$.



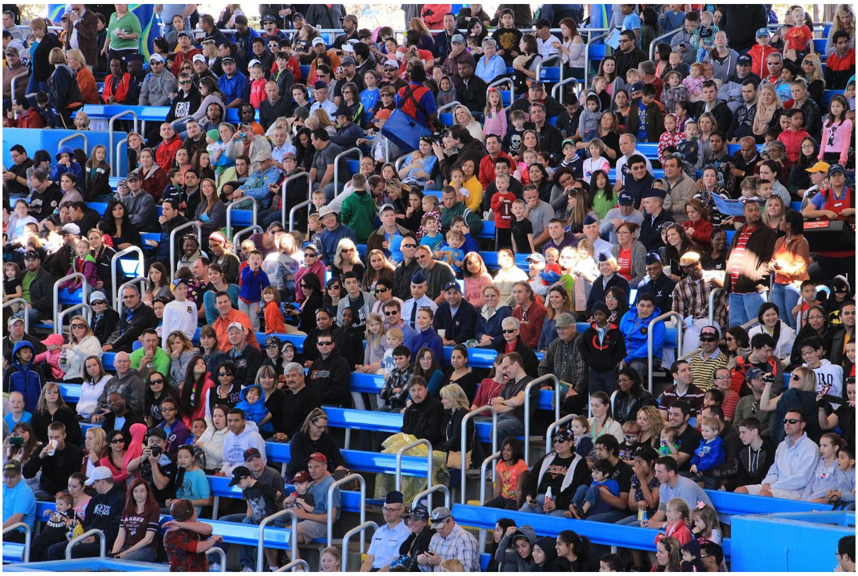
For a continuous variable, the probability $P(a < X < b)$ is the area under $f(x)$ between a and b.

Worked example. A continuous variable has $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 2$ (and 0 elsewhere). Find $E(X)$.

$$E(X) = \int_0^2 x \cdot \tfrac{1}{2}x\,dx = \int_0^2 \tfrac{1}{2}x^2\,dx = \left[\tfrac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}.$$

The **variance** uses the same idea, $\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \big(E(X)\big)^2$. For the same $f(x) = \frac{1}{2}x$ on $[0,2]$: $\int_0^2 x^2 \cdot \frac{1}{2}x\,dx = \left[\frac{x^4}{8}\right]_0^2 = 2$, so $\mathrm{Var}(X) = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9}$.

Sampling and estimation



Statistics studies a sample to learn about a whole population.

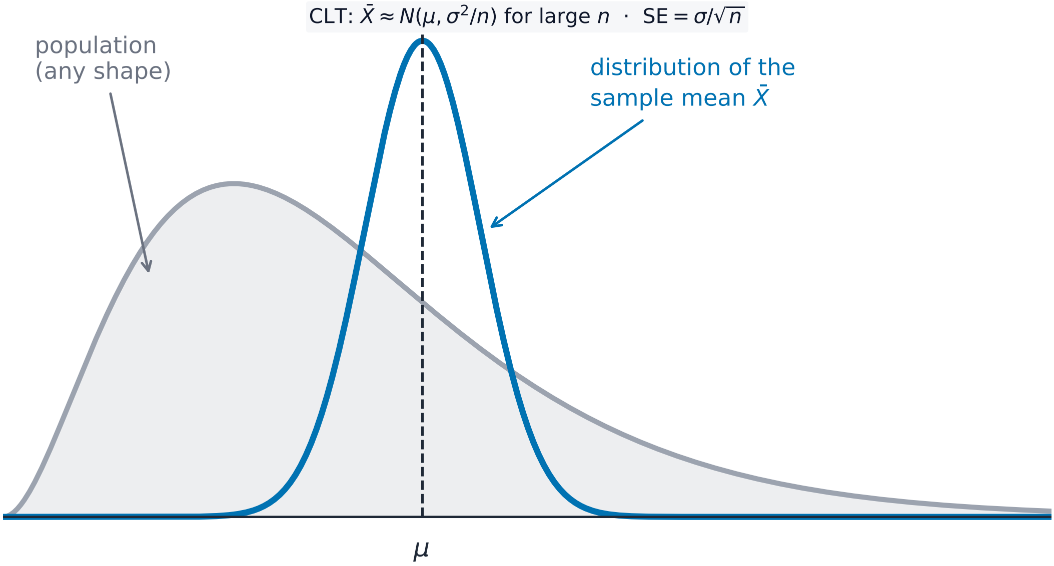
A **sample** 样本 is a small group chosen from the whole **population** 总体. A **random sample** needs **randomness** 随机性, so that every member has a fair chance of being chosen.

Some methods are **unsatisfactory** because they are **biased** 有偏: sampling only volunteers, or the first 20 people to arrive, over-represents certain kinds of people. A genuinely random sample uses **random numbers** –number every member of the population, then draw numbers (from a table or a generator) to decide who is in the sample.

The sample mean $\bar{X}$ is itself a random variable, with

$$E(\bar{X}) = \mu, \qquad \mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

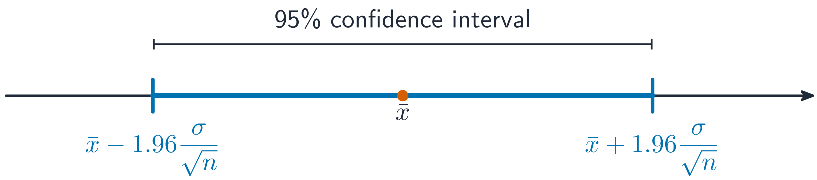
By the **Central Limit Theorem** 中心极限定理, for a large sample $\bar{X}$ is approximately normal, whatever the shape of the population.



Whatever the population's shape, the sample mean $\bar{X}$ has a narrow, near-normal distribution centred on μ.

From a sample you can find **unbiased estimates** 无偏估计 of the population mean and variance. A **confidence interval** 置信区间 gives a range that probably contains the true mean. When the population is normal with known σ (or the sample is large), a 95% interval is

$$\bar{x} \pm 1.96\,\frac{\sigma}{\sqrt{n}}.$$



A 95% confidence interval stretches 1.96 standard errors each side of the sample mean.

You can also find a confidence interval for a **population proportion** 总体比例 from a large sample.

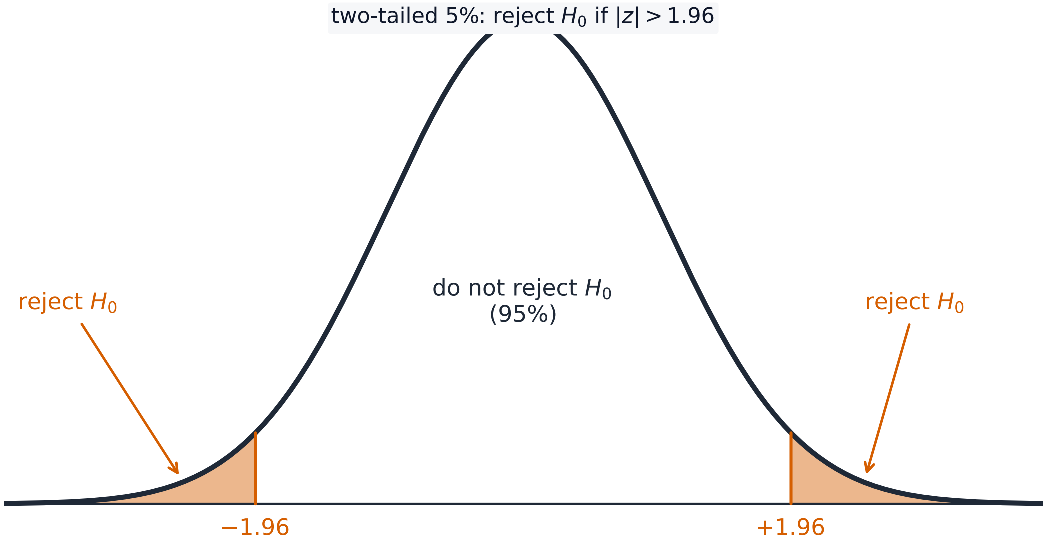
Worked example. A sample of $n = 64$ has mean $\bar{x} = 50$, from a population with $\sigma = 8$. Find a 95% confidence interval for the population mean.

$$50 \pm 1.96 \times \frac{8}{\sqrt{64}} = 50 \pm 1.96 \;\Rightarrow\; (48.0,\ 52.0).$$

Hypothesis tests

A **hypothesis test** 假设检验 uses sample data to judge a claim. You set up two statements: the **null hypothesis** 原假设 H_0 (the claim being tested, usually "no change") and the **alternative hypothesis** 备择假设 H_1 (what you suspect instead). The test is **one-tailed** 单尾 if H_1 points one way (e.g. $\mu > 50$) and **two-tailed** 双尾 if it allows both ways ($\mu \neq 50$).

You fix a **significance level** 显著性水平 (often 5%), work out a **test statistic** 检验统计量 from the data, and see whether it lands in the **rejection region** 拒绝域 (also called the **critical region**); if it does you reject H_0, otherwise the statistic is in the **acceptance region**.



A two-tailed test at 5% rejects H_0 only if the test statistic falls in a shaded tail beyond ± 1.96.

Two mistakes are possible: a **Type I error** 第一类错误 is rejecting H_0 when it is actually true; a **Type II error** 第二类错误 is accepting H_0 when it is actually false. You can find their probabilities from the rejection region: $P(\text{Type I}) = P(\text{statistic in the rejection region} \mid H_0)$ –this equals the significance level –and $P(\text{Type II}) = P(\text{statistic in the acceptance region} \mid H_1$ true for a stated value), computed from the binomial, Poisson, or normal distribution.

Worked example. A population is claimed to have mean 50, with $\sigma = 8$. A sample of $n = 64$ gives $\bar{x} = 52$. Test at the 5% level whether the mean has changed.

$H_0 : \mu = 50$ and $H_1 : \mu \neq 50$ (two-tailed). The test statistic is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{52 - 50}{8/8} = 2.$$

The critical value at 5% (two-tailed) is 1.96. Since $2 > 1.96$, you reject H_0: there is evidence the mean has changed.

Worked example (a binomial test). A coin is claimed fair but suspected of landing heads too rarely: $H_0 : p = 0.5$, $H_1 : p < 0.5$. In $n = 30$ tosses you see $X = 9$ heads. Under H_0, $X \sim B(30, 0.5)$, so the one-tailed tail probability is

$$P(X \leqslant 9) = \sum_{k=0}^{9} \binom{30}{k} (0.5)^{30} \approx 0.021.$$

Since $0.021 < 0.05$, reject H_0: the coin does seem biased against heads. (For large n the binomial is approximated by a normal; a **Poisson** test works the same way for rare events. And here, if the rule is "reject when $X \leqslant 9$", then $P(\text{Type I}) = P(X \leqslant 9 \mid p = 0.5) \approx 0.021$.)

Exam tips

- Use the **Poisson** distribution for rare, random, independent events; its mean equals its variance ($= \lambda$).
- When combining independent random variables, **variances add** (they never subtract).
- For a **hypothesis test**, state H_0 and H_1, the significance level, the test statistic, and a conclusion **in context**.
- For a **confidence interval**, use the correct z (or t) value and interpret it in words.