

Pure Mathematics 2

A-Level Mathematics

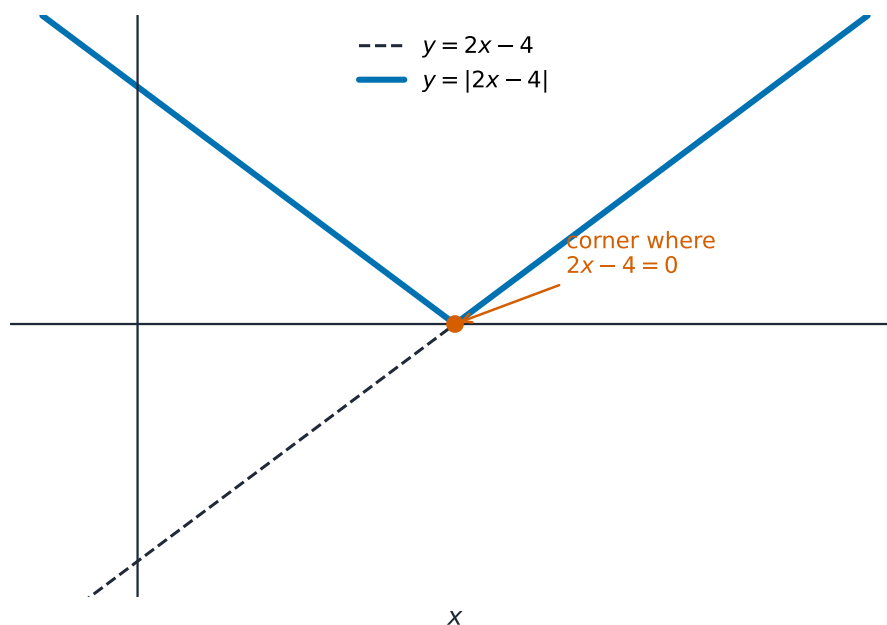
This handout covers Topic 2: **Pure Mathematics** 纯数学 2. It adds the modulus and polynomial algebra, **logarithms** 对数 and the **exponential function** 指数函数, more trigonometry, and new ways to differentiate and integrate.

Algebra

The modulus

The **modulus** 绝对值 $|x|$ is the size of a number with its sign removed, so $|x| \geq 0$ always. The graph of $y = |ax + b|$ is a "V" shape that bounces off the x -axis. Two useful rules for solving equations and inequalities are

$$|a| = |b| \Leftrightarrow a^2 = b^2, \quad |x - a| < b \Leftrightarrow a - b < x < a + b.$$



Taking the modulus folds the part of the line below the axis upward into a V.

Worked example. Solve $|3x + 8| < 9$.

Using the second rule with the inequality written as $-9 < 3x + 8 < 9$:

$$-9 < 3x + 8 < 9 \Rightarrow -17 < 3x < 1 \Rightarrow -\frac{17}{3} < x < \frac{1}{3}.$$

Polynomial division and the factor and remainder theorems

A **polynomial** 多项式 is a sum of powers of x , such as $2x^4 + 3x^2 - 5$. Its **degree** 次数 is the highest power. When you divide one polynomial by another you get a **quotient** 商 and a **remainder** 余数.

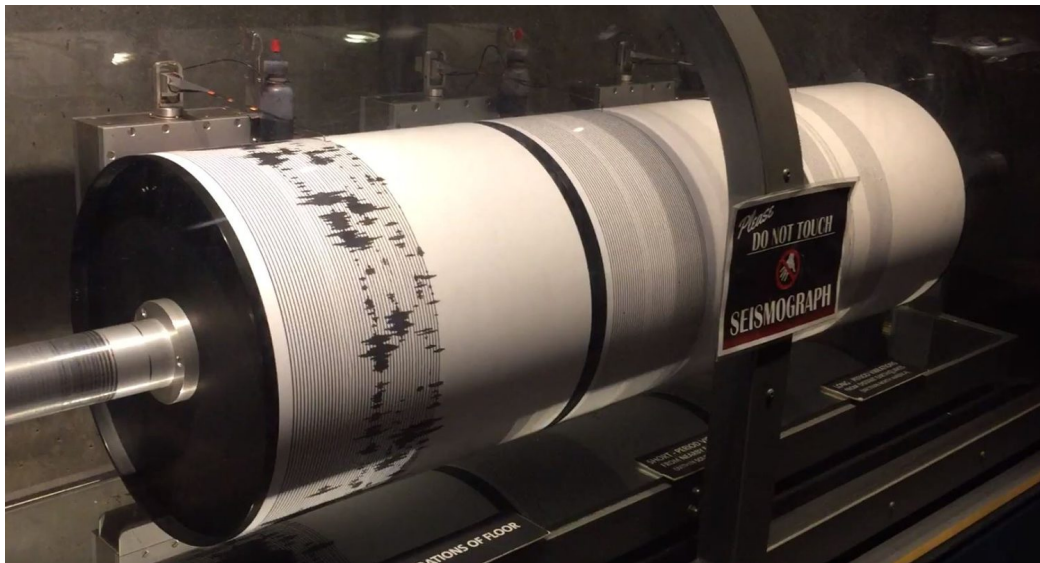
- **Remainder theorem** 余数定理: the remainder when $p(x)$ is divided by $(x - a)$ is $p(a)$.
- **Factor theorem** 因式定理: $(x - a)$ is a factor of $p(x)$ exactly when $p(a) = 0$.

Worked example. The polynomial $p(x) = 2x^4 + kx^3 + kx^2 + 17x + 18$ has factor $(x + 2)$. Find k .

By the factor theorem $p(-2) = 0$:

$$2(16) + k(-8) + k(4) + 17(-2) + 18 = 0 \Rightarrow 16 - 4k = 0 \Rightarrow k = 4.$$

Logarithmic and exponential functions



Earthquake strength is measured on the logarithmic Richter scale.

Image: Atomicdragon136, CC BY 4.0 (commons.wikimedia.org)



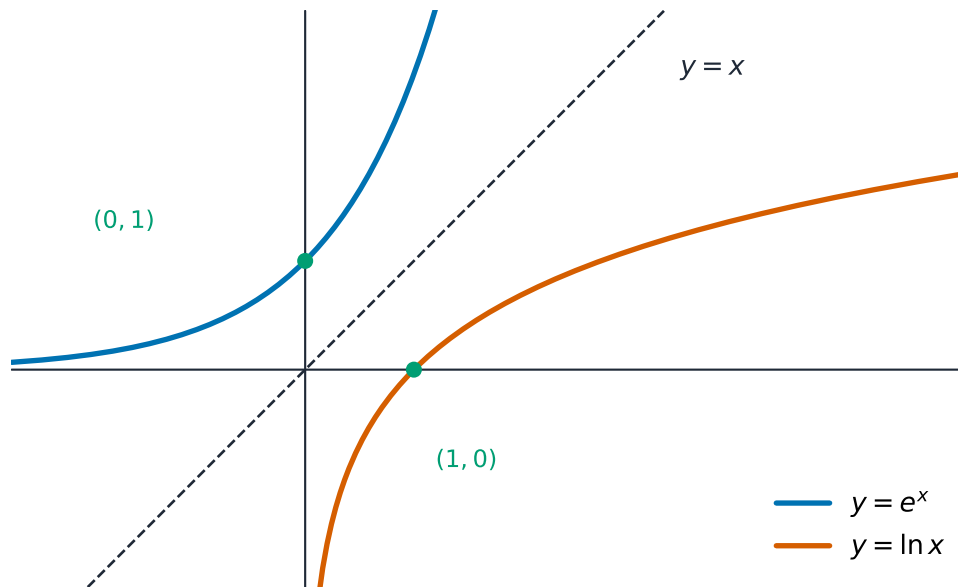
Populations can grow exponentially when resources are plentiful.

Image: Jennifer Woodard Maderazo, CC BY 2.0 (commons.wikimedia.org)

A **logarithm** answers the question "what power?". If $a^x = y$ then $x = \log_a y$. Logarithms and **indices** 指数 (powers) are reverse ideas. The **laws of logarithms** 对数定律 are

$$\log(mn) = \log m + \log n, \quad \log \frac{m}{n} = \log m - \log n, \quad \log(m^k) = k \log m.$$

The exponential function e^x and the **natural logarithm** 自然对数 $\ln x$ are inverse functions, so $\ln(e^x) = x$ and $e^{\ln x} = x$. When the unknown is in the power, take logs of both sides.



e^x and $\ln x$ undo each other, so each is the other reflected in $y = x$.

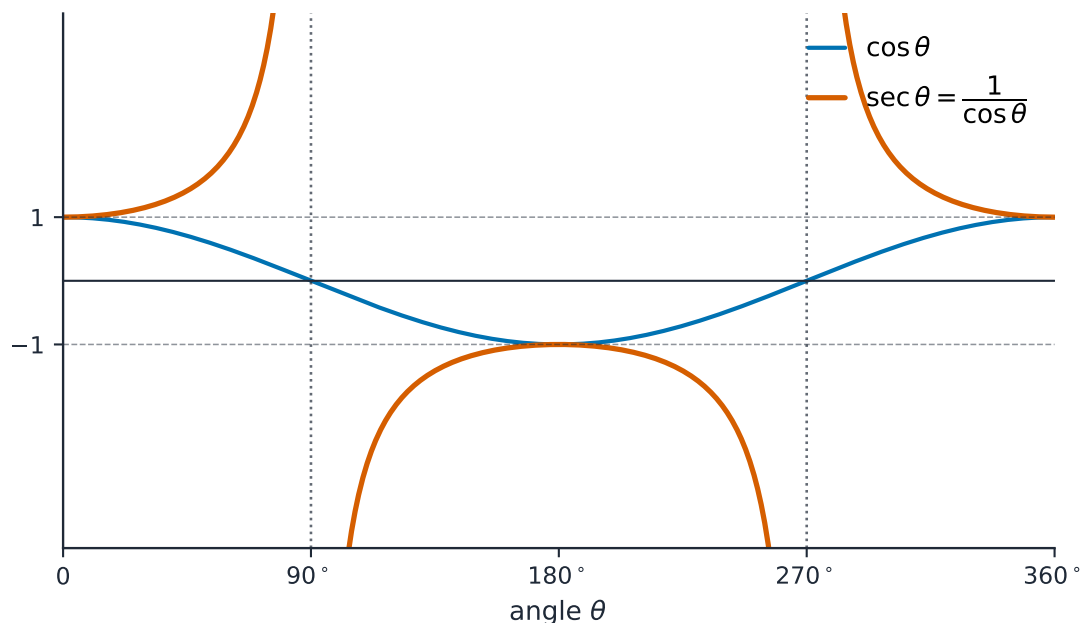
Worked example. Solve $4^x < 0.05$.

$$4^x < 0.05 \Rightarrow x \ln 4 < \ln 0.05 \Rightarrow x < \frac{\ln 0.05}{\ln 4} = -2.16 \text{ (3 s.f.)}$$

Linear form 线性形式: a relationship like $y = Ax^n$ becomes a straight line if you take logs: $\ln y = \ln A + n \ln x$. Plotting $\ln y$ against $\ln x$ gives a line with gradient n and intercept $\ln A$, so you can find the unknown constants.

Trigonometry

There are three more functions, each the reciprocal of one you know: the **secant** 正割 $\sec \theta = \frac{1}{\cos \theta}$, the **cosecant** 余割 $\csc \theta = \frac{1}{\sin \theta}$, and the **cotangent** 余切 $\cot \theta = \frac{1}{\tan \theta}$.



$\sec \theta = 1/\cos \theta$ rises to an asymptote wherever $\cos \theta = 0$.

You must know these **trigonometric identities** 三角恒等式 and choose the right one for each problem:

$$\sec^2 \theta \equiv 1 + \tan^2 \theta, \quad \csc^2 \theta \equiv 1 + \cot^2 \theta.$$

You also use the **compound angle** 复合角 formulae for $\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$, the **double angle** 二倍角 formulae

$$\sin 2A = 2 \sin A \cos A, \quad \cos 2A = 2 \cos^2 A - 1, \quad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A},$$

and the **R-formula** 辅助角公式 $a \sin \theta + b \cos \theta = R \sin(\theta + \alpha)$, where $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{b}{a}$.

Worked example. Solve $2 \tan^2 \theta + 3 \sec \theta = 18$ for $-180^\circ < \theta < 180^\circ$.

Replace $\tan^2 \theta$ with $\sec^2 \theta - 1$ to get one function:

$$2(\sec^2 \theta - 1) + 3 \sec \theta = 18 \Rightarrow 2 \sec^2 \theta + 3 \sec \theta - 20 = 0 \Rightarrow (2 \sec \theta - 5)(\sec \theta + 4) = 0.$$

So $\sec \theta = \frac{5}{2}$ or $\sec \theta = -4$, giving $\cos \theta = \frac{2}{5}$ or $\cos \theta = -\frac{1}{4}$. The solutions are $\theta = \pm 66.4^\circ$ and $\theta = \pm 104.5^\circ$.

Differentiation

Learn these standard derivatives:

$$\frac{d}{dx} e^x = e^x, \quad \frac{d}{dx} \ln x = \frac{1}{x}, \quad \frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x, \quad \frac{d}{dx} \tan x = \sec^2 x.$$

For a product or a quotient of two functions, use:

- the **product rule** 乘积法则: $(uv)' = u'v + uv'$;

- the **quotient rule** 商法则: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

When a curve is given by **parametric equations** 参数方程 $x = x(t)$, $y = y(t)$, the gradient is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. When y is defined **implicitly** 隐式 (not made the subject), differentiate every term with respect to x , using the chain rule on the y terms, then solve for $\frac{dy}{dx}$.

Worked example. Given $y = 6x \cos(x^2 + 1)$, find $\frac{dy}{dx}$.

Use the product rule with $u = 6x$ and $v = \cos(x^2 + 1)$ (and the chain rule for v):

$$\frac{dy}{dx} = 6 \cos(x^2 + 1) + 6x \cdot (-2x \sin(x^2 + 1)) = 6 \cos(x^2 + 1) - 12x^2 \sin(x^2 + 1).$$

Integration

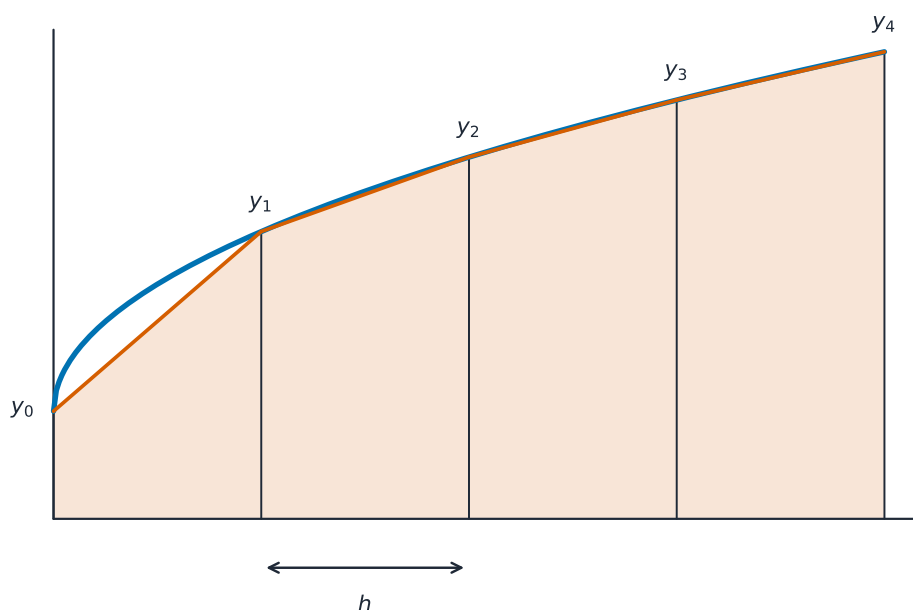
Reverse each new derivative. For a linear inside function $(ax + b)$:

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C, \quad \int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C,$$

$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + C, \quad \int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C, \quad \int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + C$$

To integrate a power of sin or cos, first use an identity to remove the power. When you cannot integrate exactly, the **trapezium rule** 梯形法则 estimates a definite integral:

$$\int_a^b y dx \approx \frac{h}{2} [y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})].$$



Each strip of width h is a trapezium; their areas add up to estimate the integral.

Worked example. Find $\int 6 \sin^2 x \, dx$.

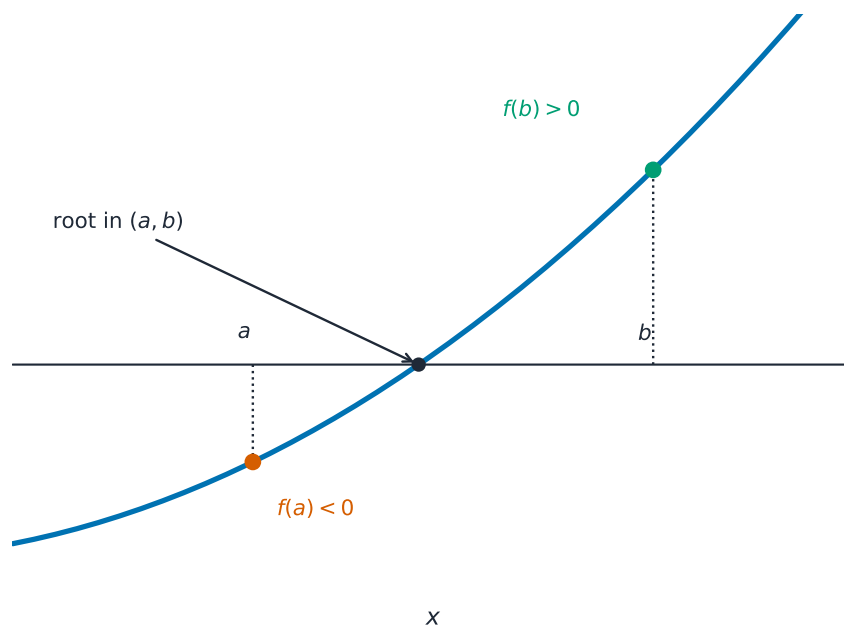
Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$:

$$\int 6 \sin^2 x \, dx = \int (3 - 3 \cos 2x) \, dx = 3x - \frac{3}{2} \sin 2x + C.$$

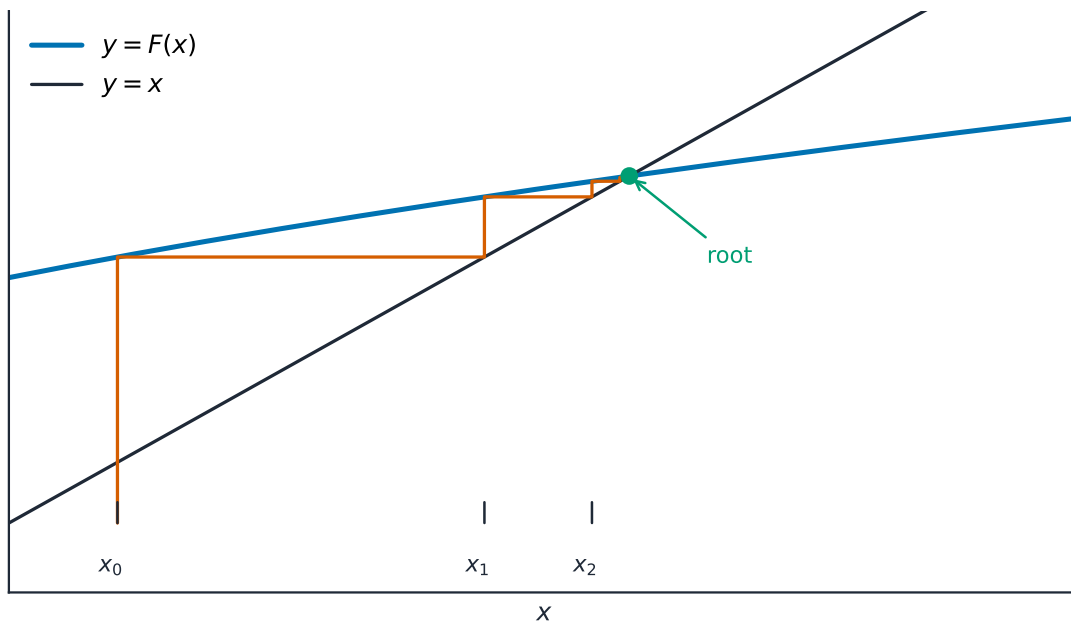
Numerical solution of equations

Many equations cannot be solved exactly. Two ideas help you find a **root** 根 (a solution).

- **Sign change** 变号: if $f(a)$ and $f(b)$ have opposite signs (and the graph has no break between them), a root lies between a and b .
- **Iteration** 迭代: rearrange the equation into the form $x = F(x)$, then use the **iterative formula** 迭代公式 $x_{n+1} = F(x_n)$. Start from a first guess x_0 and repeat. If the values settle down, they **converge** 收敛 to a root. Keep going until the answer is steady to the accuracy asked for.



$f(a)$ and $f(b)$ have opposite signs, so a root is trapped between a and b .



Each step goes up to $y = F(x)$ then across to $y = x$; the staircase closes in on the root.

Worked example. A root β of an equation satisfies $x = \sqrt[3]{-2x - 4.5}$, and $-1.4 < \beta < -1.0$. Use the iteration $x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ with $x_0 = -1.2$.

$$x_1 = \sqrt[3]{-2(-1.2) - 4.5} = \sqrt[3]{-2.1} = -1.281, \quad x_2 = \sqrt[3]{-2(-1.281) - 4.5} = -1.247, \quad \dots$$

The values settle near -1.26 , so $\beta = -1.26$ (3 s.f.).