**Week 50**

# A-Level Mathematics

Homework bundle for this week

本周作业合集

exercises.lol

## Contents 目录

A-Level Mathematics Name: ______________ Score: ______

1. For $X \sim Po(3)$, what is the mean of X?

______________

2. X has mean 5. What is $E(3X - 1)$?

______________

3. A probability density function must satisfy $\int f(x)dx$ over all x equals what value?

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4. A population has $\sigma = 8$. For a sample of $n = 64$, what is $Var(\bar{X}) = \sigma^2/n$?

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5. Population mean claimed 50, $\sigma = 8$, sample $n = 64$, $\bar{x} = 52$. What is $z = (\bar{x} - \mu)/(\sigma/\sqrt{n})$?

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6. A Type I error is:

☐ rejecting $H_0$ when it is actually true

☐ accepting $H_0$ when it is actually false

☐ choosing the wrong significance level

☐ using the wrong sample

7. In $Var(aX + b)$, the constant b:

☐ has no effect on the variance

☐ is added to the variance

☐ is squared

☐ doubles the variance

8. For a continuous variable, $P(a < X < b)$ is the area under $f(x)$ between a and b.

☐ True ☐ False

9. By the Central Limit Theorem, the sample mean is approximately normal for a large sample, whatever the population shape.

☐ True ☐ False

10. The null hypothesis $H_0$ usually states that:

☐ there is no change

- ☐ the mean has increased
- ☐ the test failed
- ☐ the sample is biased

A-Level Mathematics

1. **3**

   For a Poisson variable, the mean equals $\lambda = 3$ (and so does the variance).

2. **14**

   $E(3X - 1) = 3E(X) - 1 = 3\times5 - 1 = 14$.

3. **1**

   The total area under a pdf is always 1.

4. **1**

   $Var(\bar{X}) = \sigma^2/n = 64/64 = 1$.

5. **2**

   $z = (52 - 50)/(8/\sqrt{64}) = 2/(8/8) = 2/1 = 2$.

6. **rejecting $H_0$ when it is actually true**

   Type I = reject a true $H_0$; Type II = accept a false $H_0$.

7. **has no effect on the variance**

   Adding a constant shifts the data but does not change its spread, so b drops out.

8. **True**

   Probabilities for a continuous variable are areas under the density function.

9. **True**

   The CLT makes $\bar{X}$ approximately normal as the sample size grows.

10. **there is no change**

    $H_0$ is the claim being tested, usually "no change"; $H_1$ is the suspected alternative.

# 6.1 The Poisson distribution

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

**KEY WORDS** poisson distribution 泊松分布 • approximation 近似 • variance 方差

## Objective

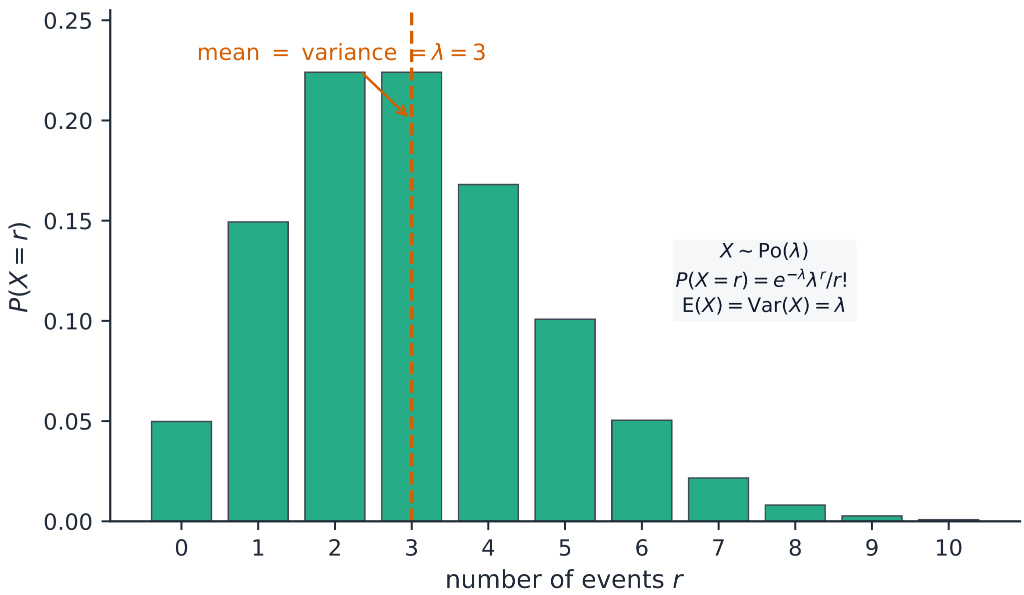
Build the skills to answer exam questions on the **Poisson distribution** 泊松分布 — using Po($\lambda$), adding Poisson variables, and the Poisson approximation to the binomial.

**You must be able to:**

- use $P(X = r) = e^{-\lambda}\dfrac{\lambda^r}{r!}$ (mean = variance = $\lambda$)
- combine independent Poisson variables (the sum is Poisson)
- use Po($np$) to approximate $B(n, p)$ when $n$ large, $p$ small

## 1 Worked examples

### Poisson probabilities




*For a Poisson variable the mean and variance both equal $\lambda$*

$X \sim \text{Po}(3)$: $P(X = 2) = e^{-3}\dfrac{3^2}{2!} = 4.5e^{-3} = 0.224.$

$P(X \geqslant 1) = 1 - P(X = 0) = 1 - e^{-3} = 0.950$.

## 2 Practice

**2.1** $X \sim \text{Po}(2)$. Find $P(X = 0)$. [2]

**2.2** State the variance of $\text{Po}(4.5)$. [1]

## 3 Exam-style questions

**3.1** For $X \sim \text{Po}(\lambda)$, the mean equals: [1]

- **A** $\lambda^2$
- **B** $\sqrt{\lambda}$
- **C** $\lambda$
- **D** $\dfrac{1}{\lambda}$

**3.2** Calls arrive at a switchboard at a mean rate of 1.5 per minute, as a Poisson process.

(a) Find the probability of exactly 2 calls in a given minute. [2]

(b) Find the probability of at least 1 call in a 2-minute period. [3]

**3.3** A factory makes items with a 0.5% defect rate. A box holds 400 items. Using a Poisson approximation, find the probability that a box contains exactly 3 defective items. [4]

## Solutions

**2.1** $P(X = 0) = e^{-2} = 0.135$.

**2.2** variance $= \lambda = 4.5$.

**3.1** **C**. For a Poisson distribution the mean is $\lambda$.

**3.2** (a) $P(X = 2) = e^{-1.5}\dfrac{1.5^2}{2!} = e^{-1.5}(1.125) = 0.251$.
(b) over 2 minutes $\lambda = 3$; $P(X \geqslant 1) = 1 - e^{-3} = 1 - 0.0498 = 0.950$.

**3.3** $\lambda = np = 400 \times 0.005 = 2$; $P(X = 3) = e^{-2}\dfrac{2^3}{3!} = e^{-2}\dfrac{8}{6}; = 0.1353 \times 1.333 = 0.180$.

**1** The random variables $X$ and $Y$ have independent distributions $X \sim \text{Po}(3)$ and $Y \sim \text{Po}(2)$ respectively.

**(a)** Find $\text{P}(2 < X < 5)$. [2]

**(b)** Find $\text{P}(X + Y > 2)$. [3]

**(c)** The total of 100 random values of $X$ and 150 random values of $Y$ is denoted by $T$.

Use a suitable approximating distribution to find $P(T < 560)$. [4]

**3** The data produced by a certain data entry firm always include a small number of incorrect characters that occur at random. The proportion of incorrect characters is denoted by $p$, and experience has shown that $p = 0.0001$. A particular data set from the firm contains 14 500 characters, of which $X$ characters are incorrect.

**(a)** Use a suitable approximating distribution to find $P(X < 4)$. [3]

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The firm's management wishes to decrease the value of $p$ by giving their employees some training. Their aim is that, for a data set containing 14 500 characters, the value of $P(X = 0)$ for the new value of $p$ should be double the value of $P(X = 0)$ when $p = 0.0001$.

**(b)** Use a suitable approximating distribution to find the new value of $p$. [3]

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**1** The number, $X$, of used computers donated to a charity has a constant average rate of 2.4 computers per week.

**(a)** State a necessary condition for $X$ to have a Poisson distribution. [1]

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Now assume that $X$ has a Poisson distribution.

**(b)** Calculate the probability that the number of computers donated during a 4-week period is more than 6 and less than 9. [3]

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**(c)** Use a suitable approximating distribution to calculate the probability that more than 50 computers are donated during a 20-week period. [4]

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**1** It is known that 1% of houses in a certain area have a wind turbine. A random sample of 400 houses in this area is chosen for a survey on domestic heating. The number of houses in the sample that have a wind turbine is denoted by $X$.

Use a suitable approximating distribution to find $\mathrm{P}(X \leqslant 3)$. [3]

**5** **(a)** The random variables $W$ and $X$ have the independent distributions $\text{Po}(1.2)$ and $\text{Po}(2.3)$ respectively.

**(i)** Find $P(3 \leqslant W + X \leqslant 5)$. [2]

The random variable $S$ is the sum of 100 independent values of $W$ and 200 independent values of $X$.

**(ii)** Use a suitable approximation to find $P(S > 600)$. [6]

**(b)** The random variable $Y$ has the distribution $\mathrm{Po}(\lambda)$, where $\lambda > 0$.

It is given that $\frac{5}{2}\mathrm{P}(Y = 3) + \mathrm{P}(Y = 4) = \mathrm{P}(Y = 5)$.

Find the value of $\lambda$. [3]

# 6.2 Linear combinations of random variables

Name: ______ Class: ______ Date: ______ **Total: 14 marks**

**KEY WORDS** expectation 期望 • variance 方差

## Objective

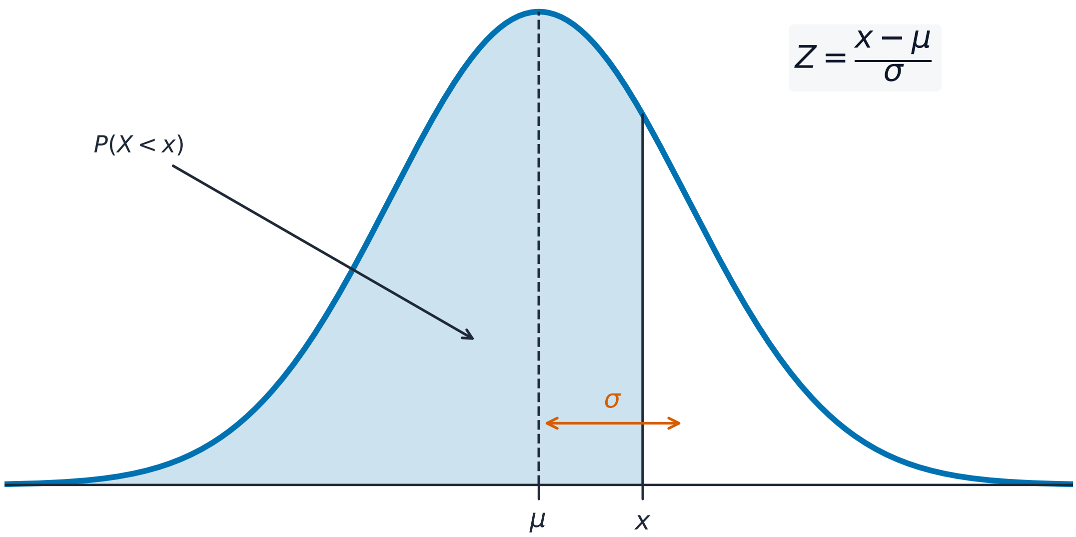
Build the skills to answer exam questions on **linear combinations of random variables 随机变量的线性组合** — the expectation and variance rules, and combining independent normal (or Poisson) variables.

**You must be able to:**

- use $E(aX+b)=aE(X)+b$, $\mathrm{Var}(aX+b)=a^2\mathrm{Var}(X)$
- use $E(aX+bY)=aE(X)+bE(Y)$, $\mathrm{Var}(aX+bY)=a^2\mathrm{Var}(X)+b^2\mathrm{Var}(Y)$
- recognise that a linear combination of independent normals is normal

## 1 Worked examples

### ■ Mean and variance rules




*If $X$ is normal then so is $aX + b$; variances add for independent variables (means always add)*

**$X$ has mean 5, variance 4:** $E(3X - 1) = 3(5) - 1 = 14$; $\text{Var}(3X - 1) = 3^2(4) = 36$.

**Watch the signs:** $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y)$ (variances **add**).

## 2 Practice

**2.1** $X$ has mean 10 and variance 9. Find $E(2X + 5)$. [2]

**2.2** For independent $X$, $Y$ with $\text{Var}(X) = 4$, $\text{Var}(Y) = 9$, find $\text{Var}(X + Y)$. [1]

## 3 Exam-style questions

**3.1** $X$ has variance 5. Then $\mathrm{Var}(4X)$ is: [1]

- **A** 20
- **B** 80
- **C** 9
- **D** 5

**3.2** Independent variables have $X \sim N(20, 3^2)$ and $Y \sim N(15, 4^2)$.

(a) Find the distribution of $X - Y$. [3]

(b) Find $P(X - Y > 2)$. [3]

**3.3** The mass of a tin is $X \sim N(500, 4^2)$ g. Four tins are packed in a box of mass $N(100, 2^2)$ g, all independent. Find the mean and standard deviation of the total mass of a packed box. [4]

## Solutions

**2.1** $E(2X+5) = 2(10) + 5 = 25$.

**2.2** $\mathrm{Var}(X+Y) = 4 + 9 = 13$.

**3.1** **B**. $\mathrm{Var}(4X) = 4^2 \times 5 = 80$.

**3.2** (a) mean $= 20 - 15 = 5$; variance $= 3^2 + 4^2 = 25$; so $X - Y \sim N(5, 25)$.
(b) $Z = \dfrac{2-5}{5} = -0.6$; $P(X - Y > 2) = P(Z > -0.6) = \Phi(0.6) = 0.7257$.

**3.3** total $T = X_1 + X_2 + X_3 + X_4 + B$; mean $= 4(500) + 100 = 2100$ g; variance $= 4(4^2) + 2^2 = 64 + 4 = 68$, so SD $= \sqrt{68} = 8.25$ g.

**1** The random variables $X$ and $Y$ have independent distributions $X \sim \mathrm{Po}(3)$ and $Y \sim \mathrm{Po}(2)$ respectively.

**(a)** Find $\mathrm{P}(2 < X < 5)$. [2]

**(b)** Find $\mathrm{P}(X + Y > 2)$. [3]

**(c)** The total of 100 random values of $X$ and 150 random values of $Y$ is denoted by $T$.

Use a suitable approximating distribution to find $P(T < 560)$. [4]

**6** The masses, in kilograms, of large and small bags of potatoes have the independent distributions $N(2.5, 0.05)$ and $N(0.8, 0.02)$ respectively.

**(a)** Find the probability that the total mass of a randomly chosen large bag of potatoes and a randomly chosen small bag of potatoes is more than 3.55 kg. [5]

**(b)** Find the probability that the mass of a randomly chosen large bag of potatoes is less than 3 times the mass of a randomly chosen small bag of potatoes. [5]

**2** The random variable $X$ has a normal distribution with mean 10 and standard deviation 3. The independent random variable $Y$ has a Poisson distribution with mean 4.

**(a)** Find the standard deviation of $X + Y$. [3]

**(b)** Find the standard deviation of $5X - Y$. [3]

**4** At an entertainment centre, the cost for using a particular video game is \$0.40 per minute. The number of minutes for which people use the video game has mean 15 and variance 9.

**(a)** Find the mean and variance of the amount people pay for using the video game. [3]

Each day, 35 people independently use the video game.

**(b)** Find the mean and variance of the total amount paid by 35 people. [3]

# 6.3 Continuous random variables

Name: ______________ Class: __________ Date: ____________ **Total: 17 marks**

**KEY WORDS** probability density function 概率密度函数 • median 中位数 • continuous random variable 连续型随机
• cumulative distribution function 累积分布函数 • percentiles 百分位数

## Objective

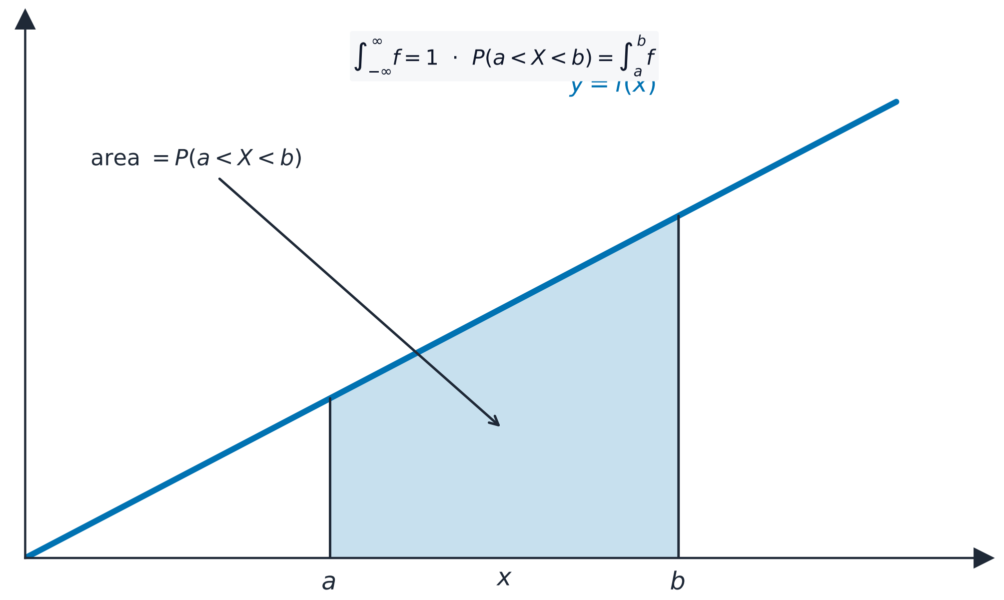
Build the skills to answer exam questions on **continuous random variables 连续型随机变量** — the **probability density function 概率密度函数** $f(x)$, finding probabilities and $E(X)$, and the **cumulative distribution function 累积分布函数** $F(x)$.

**You must be able to:**

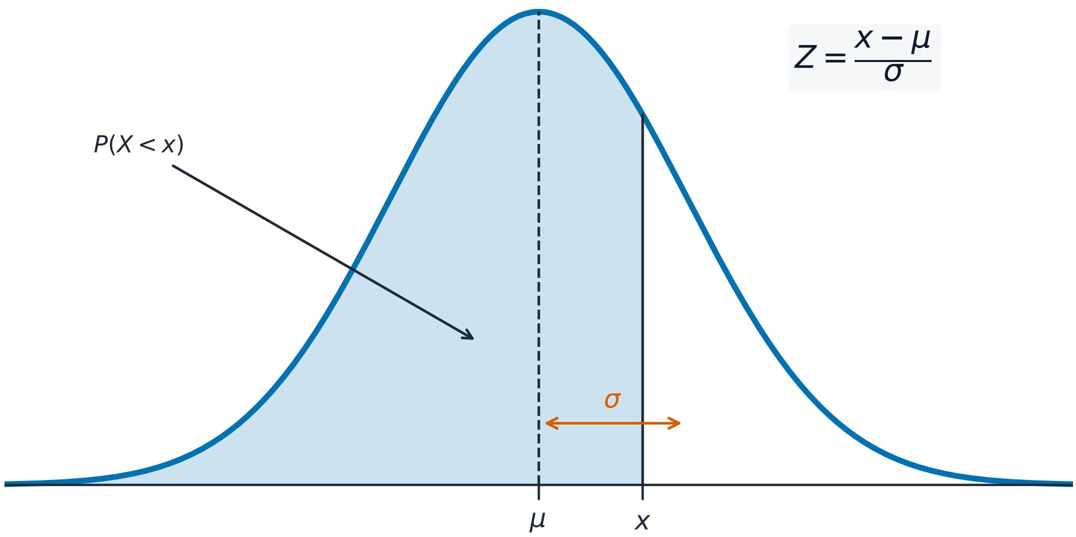
- use $\int f(x)\,dx = 1$ to find a constant; find $P(a < X < b) = \int_a^b f\,dx$
- find $E(X) = \int x f(x)\,dx$ (and $\mathrm{Var}(X)$)
- find $F(x)$ and use it for the median/percentiles

## 1 Worked examples

### ■ Density and expectation




*$P(a < X < b)$ is the area under $f(x)$ between $a$ and $b$; the total area is 1*




*The normal distribution is a continuous random variable*

$f(x) = \frac{1}{2}x$ **for** $0 \leqslant x \leqslant 2$**:** $E(X) = \displaystyle\int_0^2 x \cdot \frac{1}{2}x \, dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$.

**Median** $m$**:** solve $F(m) = 0.5$.

## 2 Practice

**2.1** $f(x) = kx$ for $0 \leqslant x \leqslant 2$. Show that $k = \frac{1}{2}$. [2]

**2.2** For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $P(X < 1)$. [2]

## 3 Exam-style questions

**3.1** For a valid pdf, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ must equal: [1]

- **A** 0
- **B** 0.5
- **C** 1
- **D** $\infty$

**3.2** A continuous random variable has $f(x) = \frac{3}{4}(1-x^2)$ for $-1 \leqslant x \leqslant 1$ (and 0 elsewhere).

(a) Verify that $\int_{-1}^{1} f(x)\,dx = 1$. [3]

(b) Find $P(X > 0.5)$. [3]

**3.3** A variable has $f(x) = \frac{3}{8}x^2$ for $0 \leqslant x \leqslant 2$.

(a) Find $E(X)$. [3]

(b) Find the median $m$. [3]

## Solutions

**2.1** $\displaystyle\int_0^2 kx\,dx = \left[\frac{kx^2}{2}\right]_0^2 = 2k = 1 \Rightarrow k = \frac{1}{2}$.

**2.2** $\displaystyle\int_0^1 \frac{1}{2}x\,dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4}$.

**3.1** **C**. The total area under a pdf is 1.

**3.2** (a) $\displaystyle\int_{-1}^1 \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1}^1 = \frac{3}{4}\left(\frac{2}{3} - (-\frac{2}{3})\right) = \frac{3}{4}\cdot\frac{4}{3} = 1$.

(b) $\displaystyle\int_{0.5}^1 \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{0.5}^1 = \frac{3}{4}\left(\frac{2}{3} - (0.5 - 0.0417)\right) = \frac{3}{4}(0.6667 - 0.4583) = 0.156$.

**3.3** (a) $\displaystyle E(X) = \int_0^2 x\cdot\frac{3}{8}x^2\,dx = \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2 = \frac{3}{8}\cdot 4 = \frac{3}{2}$.

(b) $\displaystyle F(m) = \int_0^m \frac{3}{8}x^2\,dx = \frac{m^3}{8} = 0.5 \Rightarrow m^3 = 4 \Rightarrow m = 1.59$.

**7** The time, in minutes, taken by students to complete a test is modelled by the random variable $X$ with probability density function

$$f(x) = \begin{cases} -\frac{3}{4}(x-3)(x-5) & 3 \leqslant x \leqslant 5, \\ 0 & \text{otherwise.} \end{cases}$$

**(a)** Find the probability that a randomly chosen student takes longer than 4.5 minutes to complete the test. [4]

**(b)** Write down the median of $X$. [1]

**(c)** Without performing an integration, use your answer to part **(a)** to find $P(3.5 < X < 4.5)$. [2]

## Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

**5** A random variable $X$ has probability density function given by

$$f(x) = \begin{cases} k(2x^2 - x^3) & 0 \leqslant x \leqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

**(a)** Show that $k = \frac{3}{4}$. [3]

The median of $X$ is denoted by $m$.

**(b)** **(i)** Write down the value of $\mathrm{P}(X \leqslant m)$. [1]

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**(ii)** Hence find $\mathrm{P}(\mathrm{E}(X) \leqslant X \leqslant m)$. [5]

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**7** $X$ is a random variable with probability density function given by

$$f(x) = \begin{cases} (1 + \cos \pi x) & 0 \leqslant x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

**(a)** Show that $\mathrm{P}\left(X < \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{\pi}$. [3]

**(b)** Show that $\mathrm{E}(X) = \frac{1}{2} - \frac{2}{\pi^2}$. [5]

## Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

# 6.4 Sampling and estimation

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

**KEY WORDS** sample 样本 • confidence interval 置信区间 • population 总体 • central limit theorem 中心极限定理 • sample mean 样本均值

## Objective

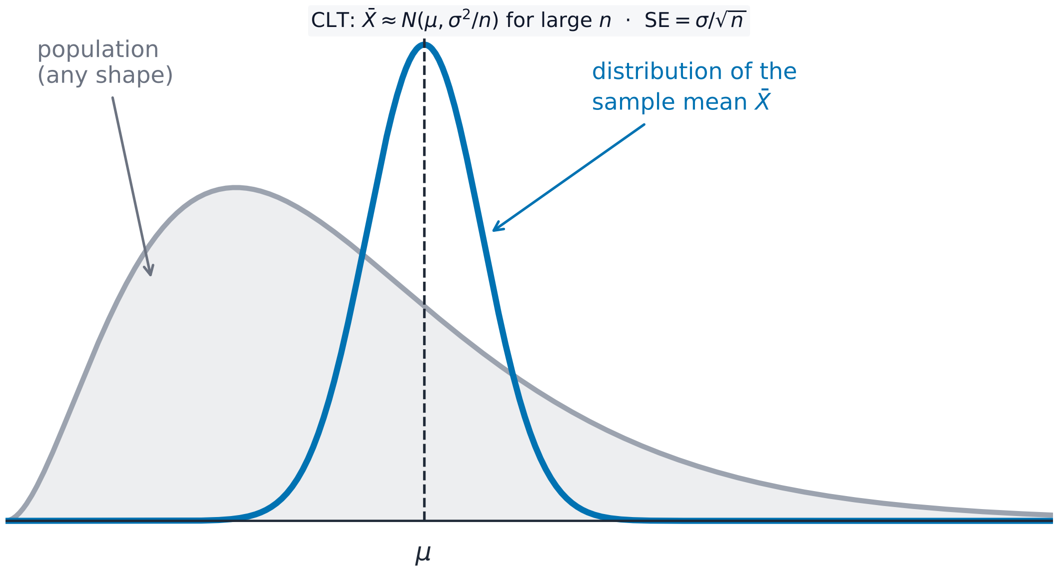
Build the skills to answer exam questions on **sampling and estimation** 抽样与估计 — the distribution of the **sample mean** 样本均值, the **Central Limit Theorem** 中心极限定理, **unbiased estimates** 无偏估计, and **confidence intervals** 置信区间.

**You must be able to:**

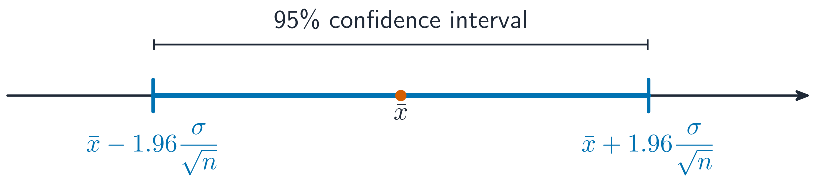
- use $E(\bar{X}) = \mu$, $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$ and the CLT
- find unbiased estimates of the population mean and variance
- find a confidence interval $\bar{x} \pm z\dfrac{\sigma}{\sqrt{n}}$

## 1 Worked examples

### Confidence interval




*By the CLT, $\bar{X}$ is approximately normal for large $n$, centred on $\mu$ with variance $\sigma^2/n$*




*A 95% interval reaches 1.96 standard errors each side of $\bar{x}$*

$n = 64$**,** $\bar{x} = 50$**,** $\sigma = 8$**:** $50 \pm 1.96\frac{8}{8} = 50 \pm 1.96 = (48.0,\ 52.0)$.

## 2 Practice

**2.1** A population has $\sigma = 10$. For a sample of $n = 25$, find the standard error $\frac{\sigma}{\sqrt{n}}$. [2]

**2.2** State the value of $z$ used in a 95% confidence interval. [1]

## 3 Exam-style questions

**3.1** As the sample size $n$ increases, $\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$: [1]

- **A** increases
- **B** decreases
- **C** stays the same
- **D** becomes negative

**3.2** A random sample of 100 light bulbs has mean lifetime $\bar{x} = 1200$ hours. The population standard deviation is $\sigma = 150$ hours.

(a) Find a 95% confidence interval for the mean lifetime. [3]

(b) State what changes to the interval if a 99% level ($z = 2.576$) is used instead. [2]

**3.3** A sample of 8 values gives $\sum x = 120$ and $\sum x^2 = 1840$. Find unbiased estimates of the population mean and variance. [4]

## Solutions

**2.1** $\dfrac{10}{\sqrt{25}} = \dfrac{10}{5} = 2.$

**2.2** 1.96.

**3.1** **B**. Dividing by a larger $n$ makes $\mathrm{Var}(\bar{X})$ smaller.

**3.2** (a) $1200 \pm 1.96\dfrac{150}{\sqrt{100}} = 1200 \pm 1.96(15) = 1200 \pm 29.4$; (1170.6, 1229.4).
(b) the interval becomes **wider** (a higher confidence level needs a larger $z$, here 2.576).

**3.3** mean $\bar{x} = \dfrac{120}{8} = 15$; unbiased variance $s^2 = \dfrac{1}{n-1}\left(\sum x^2 - \dfrac{(\sum x)^2}{n}\right) = \dfrac{1}{7}\left(1840 - \dfrac{120^2}{8}\right) = \dfrac{1}{7}(1840 - 1800) = \dfrac{40}{7} = 5.71.$

**4** The masses of a certain species of animal are known to be normally distributed with standard deviation $\sigma$ kg. A researcher obtains the masses of a random sample of $n$ animals of this species and uses these masses to find two confidence intervals ($\alpha$% and 90%) for the population mean. The width of the $\alpha$% confidence interval is $1.414 \times$ the width of the 90% confidence interval.

**(a)** Find $\alpha$. [3]

**(b)** Find the probability that the 90% confidence interval contains the population mean given that the $\alpha$% confidence interval contains the population mean. [1]

**3** The times, in minutes, taken by students to complete a test have mean $\mu$ and standard deviation $\sigma$. The times taken by a random sample of 100 students are noted and are used to calculate a 95% confidence interval for $\mu$.

**(a)** Given that the end points of the 95% confidence interval are 31.02 and 33.98, correct to 4 significant figures, calculate the value of $\sigma$. [3]

**(b)** The calculation of the confidence interval required the use of the Central Limit theorem.

Explain why it is valid to use the Central Limit theorem in this case. [1]

**(c)** A researcher calculates a number, $r$, of 95% confidence interval for $\mu$.

Find the largest value of $r$ such that the probability that all $r$ confidence intervals contain the true value of $\mu$ is greater than 0.5. [4]

**2** The random variable $X$ has the distribution $\mathrm{B}\left(8, \frac{3}{4}\right)$. A random sample of 100 values of $X$ is chosen, and the sample mean, $\overline{X}$, is found.

**(a)** Find $\mathrm{P}(\overline{X} > 6.2)$. You are **not** expected to use a continuity correction. [6]

**(b)** State why the Central Limit Theorem was needed in the calculation in part **(a)**. [1]

**3** The time, $T$ minutes, for a certain daily bus journey is normally distributed. The bus company claims that the mean of $T$ is 45. A passenger believes that the mean of $T$ is actually greater than 45. She notes the times taken for this journey on a random sample of 60 days. The results are summarised below.

$$n = 60 \qquad \sum t = 2750 \qquad \sum t^2 = 127000$$

**(a)** Calculate unbiased estimates of the population mean and variance. [3]

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**(b)** Test the passenger's belief at the 5% significance level. [5]

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**6** A manufacturer of cell phones claims that 25% of students own a Pumpkin phone. Jeyeraj thinks that the proportion of students at his large college who own a Pumpkin phone is less than 25%. He plans to test the manufacturer's claim. He chooses a random sample of 30 students at his college. If the number of students who own a Pumpkin phone is less than 5, Jeyeraj will reject the manufacturer's claim.

**(a)** State suitable hypotheses for the test. [1]

**(b)** Given that the true proportion of students at the college who own a Pumpkin phone is 10%, use a binomial distribution to find the probability of a Type II error. [3]

At Florence's college, in a random sample of 40 students, it was found that 5 own a Pumpkin phone.

**(c)** Calculate an approximate 95% confidence interval for the proportion of students at Florence's college who own a Pumpkin phone. [3]

A-LEVEL MATHEMATICS • EXERCISE SHEET

# 6.5 Hypothesis tests

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

**KEY WORDS** significance level 显著性水平 • type ii error 第二类错误 • type i error 第一类错误 • rejection region 拒
• two-tailed 双尾

## Objective

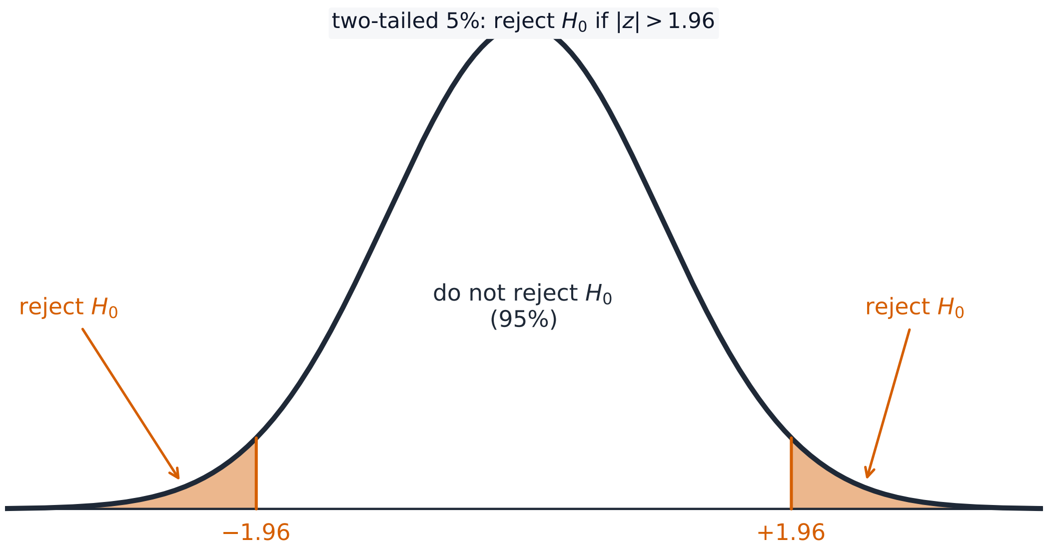
Build the skills to answer exam questions on **hypothesis tests 假设检验** — stating $H_0$ and $H_1$, one- and two-tailed tests, the **test statistic 检验统计量** and **rejection region 拒绝域**, and **Type I/II errors 第一/二类错误**.

**You must be able to:**

- state $H_0$, $H_1$ and choose one- or two-tailed
- compute a $z$ test statistic and compare with the critical value
- explain Type I and Type II errors

## 1 Worked examples

### A two-tailed test




*A two-tailed test at 5% rejects $H_0$ only if the statistic falls beyond $\pm 1.96$*

**Claim** $\mu = 50$, $\sigma = 8$, $n = 64$, $\bar{x} = 52$, **test at** $5\%$: $H_0 : \mu = 50$, $H_1 : \mu \neq 50$.

$z = \dfrac{52 - 50}{8/8} = 2$. Since $2 > 1.96$, **reject** $H_0$.

## 2 Practice

**2.1** State the meaning of a **Type I error**. [1]

**2.2** Give the critical $z$ value for a one-tailed test at the 5% level. [1]

## 3 Exam-style questions

**3.1** In a test at the 5% level, $H_0$ is rejected when the $p$-value is: [1]

- **A** greater than 0.05
- **B** less than 0.05
- **C** equal to 1
- **D** negative

**3.2** A machine should fill bottles to a mean of 500 ml with $\sigma = 6$ ml. A sample of 36 bottles has mean $\bar{x} = 502$ ml. Test at the 5% level whether the mean has increased.

(a) State $H_0$ and $H_1$, and whether the test is one- or two-tailed. [2]

(b) Carry out the test and state your conclusion. [4]

**3.3** A coin is tossed 20 times to test $H_0 : p = 0.5$ against $H_1 : p > 0.5$ at the 5% level, using the number of heads $X \sim B(20, 0.5)$. The critical region is $X \geqslant 15$.

(a) Find $P(X \geqslant 15)$ under $H_0$ and explain why $X \geqslant 15$ is used. [3]

(b) State what a Type I error would mean in this context. [1]

## Solutions

**2.1** rejecting $H_0$ when it is in fact true.

**2.2** 1.645.

**3.1** **B**. Reject $H_0$ when the $p$-value is below the significance level 0.05.

**3.2** (a) $H_0\colon \mu = 500$, $H_1\colon \mu > 500$; one-tailed.
(b) $z = \dfrac{502 - 500}{6/\sqrt{36}} = \dfrac{2}{1} = 2$; critical value 1.645; since $2 > 1.645$, reject $H_0$: there is evidence the mean has increased.

**3.3** (a) $P(X \geqslant 15) = 0.0207$ (from tables); this is the smallest "tail" region with probability $\leqslant 0.05$, so it is the 5% critical region.
(b) concluding the coin is biased towards heads ($p > 0.5$) when in fact it is fair.

**2** The mean mass of packets of Trueleaf tea is supposed to be 500 grams. An inspector wishes to test whether this value is correct. He weighs 60 randomly chosen packets and notes the mass, $x$ grams, of each packet. The results are summarised as follows.

$$n = 60 \qquad \sum x = 29\,970 \qquad \sum x^2 = 14\,970\,300$$

Test, at the 5% significance level, whether the population mean mass is 500 grams. [8]

**5** It is known that 20% of households in a certain country contain more than 4 people. Laxmi believes that, in her town, the percentage is lower than 20%. She chooses a random sample of 40 households in her town and notes the number which contain more than 4 people. She then carries out a test at the 2.5% significance level using a binomial distribution.

**(a)** Find the probability of a Type I error. [4]

**(b)** State the rejection region for the test. [1]

Laxmi finds that exactly 2 households in her sample contain more than 4 people.

**(c)** Explain why it is impossible for Laxmi to make a Type II error. [1]

**4** An inspector believes that 18% of cups made at a certain factory contain flaws. The factory owner claims that the true percentage is less than 18%. The inspector examines a random sample of 40 cups and finds that 3 of them contain flaws.

**(a)** Stating a necessary assumption, use a binomial distribution to test the factory owner's claim at the 5% significance level. [6]

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**(b)** Explain why it would **not** be appropriate to use the Poisson approximation to the binomial distribution to carry out the test in part **(a)**. [1]

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**6** The weekly profit, in dollars, made by a certain firm has a normal distribution. In the past, the weekly profit had the distribution $N(736, 26^2)$. Following a change in management, the mean weekly profit for 35 randomly chosen weeks is \$725.

**(a)** Stating a necessary assumption, test at the 2% significance level whether the mean weekly profit has decreased. [6]

The mean weekly profit for another random sample of 35 weeks is found and a similar test is carried out at the 2% significance level.

**(b)** State the probability of a Type I error. [1]

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**(c)** Given that the mean weekly profit is now in fact \$718, find the probability of a Type II error. [5]

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## Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

**3** The time, $T$ minutes, for a certain daily bus journey is normally distributed. The bus company claims that the mean of $T$ is 45. A passenger believes that the mean of $T$ is actually greater than 45. She notes the times taken for this journey on a random sample of 60 days. The results are summarised below.

$$n = 60 \qquad \Sigma t = 2750 \qquad \Sigma t^2 = 127\,000$$

**(a)** Calculate unbiased estimates of the population mean and variance. [3]

**(b)** Test the passenger's belief at the 5% significance level. [5]

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