Week 12

A-Level Mathematics

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

A-Level Mathematics Name: ________________ Score: ______

1. Solve $|x - 2| < 3$. The solution is $-1 < x < ?$ What is the upper bound?

   ____

2. Evaluate $\log_2(8)$.

   ____

3. The secant function $\sec\theta$ is equal to:

   ☐ $1 / \cos\theta$

   ☐ $1 / \sin\theta$

   ☐ $1 / \tan\theta$

   ☐ $\cos\theta / \sin\theta$

4. The derivative of $\ln x$ is $1/x$. What is its value at $x = 4$?

   ____

5. What is $\int e^x\,dx$?

   ☐ $e^x + C$

   ☐ $x\,e^x + C$

   ☐ $e^{x+1} + C$

   ☐ $\ln x + C$

6. If f(a) and f(b) have opposite signs (and f is continuous between them), then between a and b there is:

   ☐ at least one root

   ☐ no root

   ☐ a maximum only

   ☐ always exactly two roots

7. By the remainder theorem, what is the remainder when $p(x) = x^2 + 3x + 2$ is divided by $(x - 1)$?

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8. What is $\ln(e^5)$?

   ____

9. What is $\int (1/x)\,dx$?

- ☐ $\ln|x| + C$
- ☐ $1/x^2 + C$
- ☐ $x^{-2} + C$
- ☐ $e^x + C$

10. An iterative formula has the form:
    - ☐ x_{n+1} = F(x_n)
    - ☐ x = a + b
    - ☐ f(x) = 0 only
    - ☐ x_n = n^2

A-Level Mathematics

1. **5**

   $|x-2| < 3$ means $2-3 < x < 2+3$, i.e. $-1 < x < 5$.

2. **3**

   $2^3 = 8$, so $\log_2(8) = 3$.

3. **1 / cos θ**

   $\sec\theta = 1/\cos\theta$; $\csc\theta = 1/\sin\theta$; $\cot\theta = 1/\tan\theta$.

4. **0.25**

   $d/dx(\ln x) = 1/x = 1/4 = 0.25$.

5. **$e^x + C$**

   e^x is its own integral (and derivative): $\int e^x\, dx = e^x + C$.

6. **at least one root**

   A sign change of a continuous function guarantees a root between a and b.

7. **6**

   Remainder $= p(1) = 1 + 3 + 2 = 6$.

8. **5**

   ln and e are inverses, so $\ln(e^5) = 5$.

9. **$\ln|x| + C$**

   $\int(1/x)\, dx = \ln|x| + C$ (the power rule fails for $n = -1$).

10. **x_{n+1} = F(x_n)**

    Iteration repeatedly applies x_{n+1} = F(x_n) from a first guess.

A-LEVEL MATHEMATICS • EXERCISE SHEET

2.1 Algebra (modulus, factor and remainder theorems)

Name: ______________ Class: __________ Date: ____________ **Total: 16 marks**

KEY WORDS factor 因式定理 • remainder 余数 • remainder theorem 余数定理 • polynomial 多项式 • modulus 绝对值

Objective

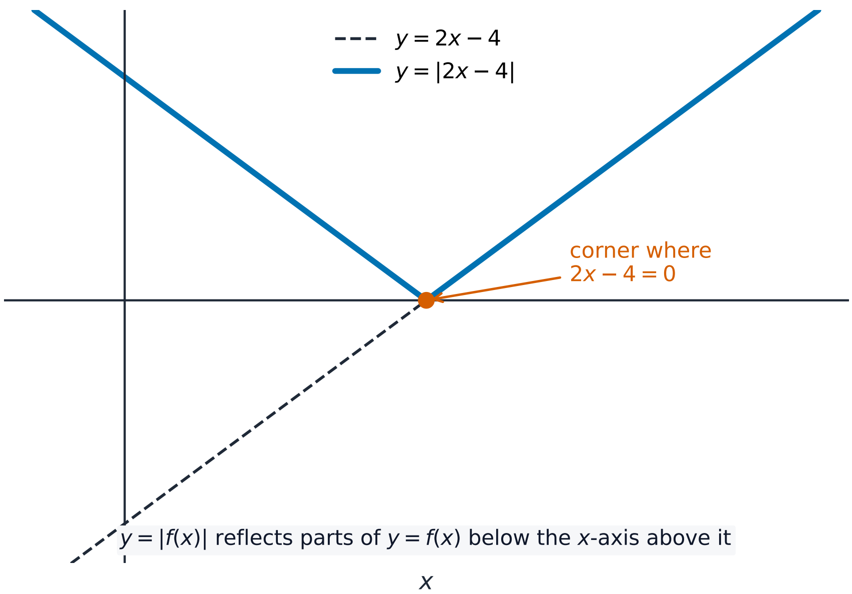
Build the skills to answer exam questions on **algebra** 代数 — the **modulus** 绝对值, and the **factor** 因式定理 and **remainder theorems** 余数定理 for polynomials.

You must be able to:

- solve modulus equations and inequalities ($|x-a|<b$, $|a|=|b| \Leftrightarrow a^2=b^2$)
- use the remainder theorem (remainder $= p(a)$)
- use the factor theorem to factorise a polynomial

1 Worked examples

■ Modulus & factor theorem



Taking the modulus folds the part below the axis up into a "V"

Solve $|3x+8|<9$. $-9<3x+8<9 \Rightarrow -17<3x<1 \Rightarrow -\frac{17}{3}<x<\frac{1}{3}$.

$p(x) = 2x^4 + kx^3 + kx^2 + 17x + 18$ **has factor** $(x + 2)$**. Find** k**.** $p(-2) = 0$: $32 - 8k + 4k - 34 + 18 = 0 \Rightarrow 16 - 4k = 0 \Rightarrow k = 4$.

2 Practice

2.1 Solve $|2x - 1| = 7$. [2]

2.2 Find the remainder when $x^3 - 2x + 5$ is divided by $(x - 2)$. [2]

3 Exam-style questions

3.1 $(x - 3)$ is a factor of $x^3 - 4x^2 + x + c$. The value of c is: [1]

- **A** -6
- **B** 6
- **C** 12
- **D** -12

3.2 Solve the inequality $|x + 2| > |x - 4|$. [3]

3.3 The polynomial $p(x) = x^3 + ax^2 + bx + 6$ has a factor $(x - 1)$ and leaves a remainder of 8 when divided by $(x + 1)$.

(a) Form two equations in a and b and solve them. [5]

(b) Hence factorise $p(x)$ completely. [3]

Solutions

2.1 $2x - 1 = \pm 7$; $x = 4$ or $x = -3$.

2.2 $p(2) = 8 - 4 + 5 = 9$.

3.1 **B**. $p(3) = 27 - 36 + 3 + c = 0 \Rightarrow c = 6$.

3.2 square both sides: $(x+2)^2 > (x-4)^2$; $x^2 + 4x + 4 > x^2 - 8x + 16 \Rightarrow 12x > 12$; $x > 1$.

3.3 (a) factor $(x - 1)$: $p(1) = 1 + a + b + 6 = 0 \Rightarrow a + b = -7$; remainder $p(-1) = -1 + a - b + 6 = 8 \Rightarrow a - b = 3$; add the equations: $2a = -4$, so $a = -2$, $b = -5$.
(b) $p(x) = x^3 - 2x^2 - 5x + 6$; divide by $(x-1)$ to get $x^2 - x - 6$; $x^2 - x - 6 = (x-3)(x+2)$; so $p(x) = (x - 1)(x - 3)(x + 2)$.

3 **(a)** Solve the equation $|2x-3| = |5x+2|$. [3]

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(b) Hence solve the equation $|2\sec\theta - 3| = |5\sec\theta + 2|$ for $\pi < \theta < 2\pi$. Give your answer correct to 3 significant figures. [3]

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7 The polynomial $\mathrm{p}(x)$ is defined by

$$\mathrm{p}(x) = 2x^4 + kx^3 + kx^2 + 17x + 18,$$

where k is a constant. It is given that $(x+2)$ is a factor of $\mathrm{p}(x)$.

(a) Find the value of k. [2]

It is given that the equation $\mathrm{p}(x) = 0$ has exactly two real roots, denoted by α and β, where α is an integer and β is not an integer.

(b) State the value of α and show that β satisfies the equation $x = \sqrt[3]{-2x-4.5}$. [4]

(c) Show by calculation that $-1.4 < \beta < -1.0$. [2]

(d) Use an iterative formula, based on the equation in part **(b)**, to find the value of β correct to 3 significant figures. Give the result of each iteration to 5 significant figures. [3]

3 **(a)** Solve the inequality $|3x-4| \leqslant |2x+5|$. [4]

(b) Hence find the largest integer N satisfying the inequality $|3 \times 7^{0.01N} - 4| \leqslant |2 \times 7^{0.01N} + 5|$. [3]

4 The polynomial $\mathrm{p}(x)$ is defined by

$$\mathrm{p}(x) = x^4 - 10x^3 + 20x^2 - 30x + 40.$$

(a) Find the quotient when $\mathrm{p}(x)$ is divided by $(x^2 + 3)$ and show that the remainder is -11. [3]

(b) Hence find the real roots of the equation $\mathrm{p}(x) + 11 = 0$. Give your answers in exact form. [3]

2 **(a)** Use logarithms to solve the inequality $4^x < 0.05$. Give your answer in the form $x < a$, where the value of a is correct to 3 significant figures. [2]

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(b) Solve the inequality $|3x+8| < 9$. [3]

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(c) Hence state the integers that satisfy both of the inequalities in parts **(a)** and **(b)**. [1]

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5 The polynomial $p(x)$ is defined by

$$p(x) = ax^3 + bx^2 - ax - 24,$$

where a and b are constants. It is given that $(2x-3)$ is a factor of $p(x)$ and that the remainder is -15 when $p(x)$ is divided by $(x+1)$.

(a) Find the values of a and b. [4]

(b) Hence factorise $\mathrm{p}(x)$ completely. [3]

(c) Hence solve the equation $\mathrm{p}(3\operatorname{cosec}\theta) = 0$ for $90° < \theta < 270°$. [2]

2 **(a)** Sketch on the same diagram the graphs of $y = |2x - 9|$ and $y = 4x - 5$. [2]

(b) Solve the inequality $|2x - 9| < 4x - 5$. [3]

5 The polynomial $p(x)$ is defined by

$$p(x) = ax^4 + bx^3 + 13x^2 - 35x + 15,$$

where a and b are constants. It is given that $(2x - 1)$ and $(x - 3)$ are factors of $p(x)$.

(a) Find the values of a and b. [4]

(b) Hence factorise $p(x)$. [3]

(c) Find the least positive value of θ in radians such that $p(\cot 2\theta) = 0$. [2]

2.2 Logarithmic and exponential functions

Name: ______ Class: ______ Date: ______ **Total: 13 marks**

KEY WORDS logarithmic and exponential functions 对数与指数函数 • laws of logarithms 对数定律 • logarithms 对数 • exponential function 指数函数 • linear form 线性形式

Objective

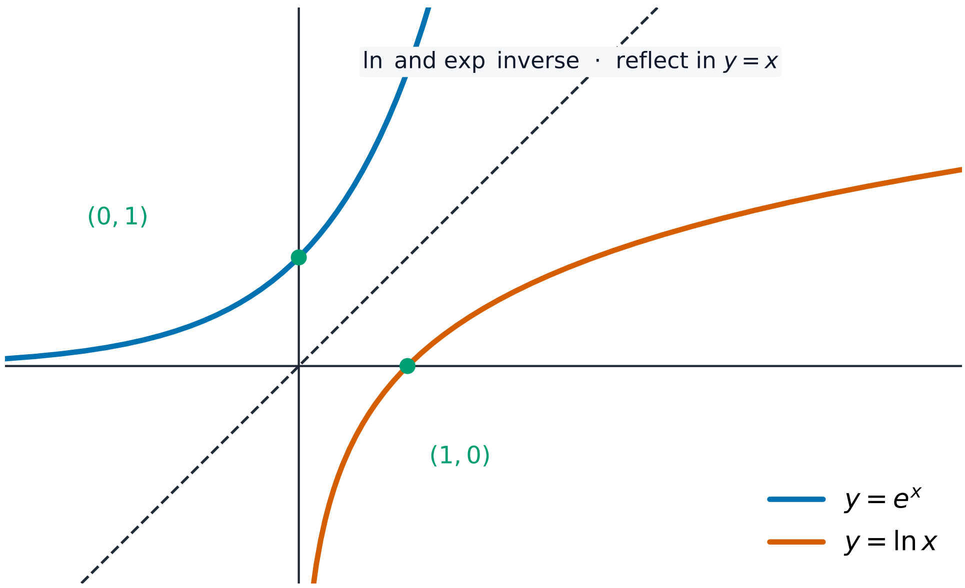
Build the skills to answer exam questions on **logarithmic and exponential functions** 对数与指数函数 — the laws of logs, solving equations with the unknown in the power, and reducing $y = Ax^n$ to **linear form** 线性形式.

You must be able to:

- use the laws of logarithms and $\ln / e^x$ as inverses
- solve equations such as $a^x = b$ by taking logs
- use a log–log (or log–linear) plot to find unknown constants

1 Worked examples

■ Solving and linearising



e^x and $\ln x$ undo each other, so each is the other reflected in $y = x$

Solve $4^x < 0.05$**.** $x \ln 4 < \ln 0.05 \Rightarrow x < \dfrac{\ln 0.05}{\ln 4} = -2.16$ (3 s.f.).

Linear form: $y = Ax^n \Rightarrow \ln y = \ln A + n \ln x$ — plot $\ln y$ against $\ln x$: gradient n, intercept $\ln A$.

2 Practice

2.1 Write $\log 12$ in terms of $\log 2$ and $\log 3$. [2]

2.2 Solve $e^{2x} = 7$, giving x to 3 s.f. [2]

3 Exam-style questions

3.1 The solution of $3^x = 20$ is $x =$ [1]

- **A** $\dfrac{\ln 20}{\ln 3}$
- **B** $\dfrac{\ln 3}{\ln 20}$
- **C** $\ln \dfrac{20}{3}$

- **D** $20 - 3$

3.2 Solve the equation $\ln(2x + 1) + \ln 3 = \ln(4x + 9)$. [3]

3.3 The variables x and y satisfy $y = Ax^n$. When $\ln y$ is plotted against $\ln x$, a straight line of gradient 1.5 passing through (0, 2) is obtained.

(a) Find the values of n and A. [3]

(b) Find y when $x = 4$. [2]

Solutions

2.1 $12 = 2^2 \times 3$, so $\log 12 = 2\log 2 + \log 3$.

2.2 $2x = \ln 7 \Rightarrow x = \frac{1}{2}\ln 7 = 0.973$.

3.1 **A**. $x \ln 3 = \ln 20 \Rightarrow x = \dfrac{\ln 20}{\ln 3}$.

3.2 $\ln(3(2x+1)) = \ln(4x+9)$; $6x + 3 = 4x + 9 \Rightarrow 2x = 6$; $x = 3$.

3.3 (a) gradient $= n = 1.5$; intercept $\ln A = 2 \Rightarrow A = e^2 = 7.39$.
(b) $y = 7.39 \times 4^{1.5} = 7.39 \times 8 = 59.1$.

2 Solve the equation $e^{2x}\left(e^{2x}-8\right)=48$. [3]

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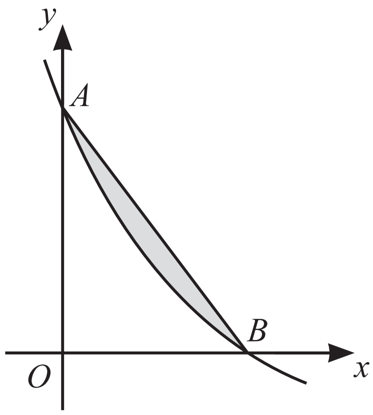

The diagram shows the curve with equation $y = 8e^{-\frac{1}{2}x} - 1$. The curve meets the axes at the points A and B. The shaded region is bounded by the curve and the line segment AB.

(a) Show that the x-coordinate of B is $6\ln 2$. [2]

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(b) Find the area of the shaded region. Give your answer in the form $p\ln 2 - q$, where p and q are positive integers. [5]

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1 Solve the equation $\ln(3x+5)-\ln(x-2)=4$. Give your answer in an exact form. [4]

2 **(a)** Use logarithms to solve the inequality $4^x < 0.05$. Give your answer in the form $x < a$, where the value of a is correct to 3 significant figures. [2]

(b) Solve the inequality $|3x+8| < 9$. [3]

(c) Hence state the integers that satisfy both of the inequalities in parts **(a)** and **(b)**. [1]

6 A curve has equation $(x^2 - 3)\ln y + 6x = 14$.

(a) Show that there is no point on the curve at which the y-coordinate is e^{-1}. [3]

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(b) Find the equation of the tangent to the curve at the point $(2,\ e^2)$. Give your answer in the form $y = mx + c$, where m and c are exact constants. [6]

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2.3 Trigonometry (reciprocal, compound, double angle, R-formula)

Name: ______________ Class: __________ Date: ____________ **Total: 12 marks**

KEY WORDS r-formula 辅助角公式 • compound 复合角 • double angle 二倍角 • further trigonometry 三角学

Objective

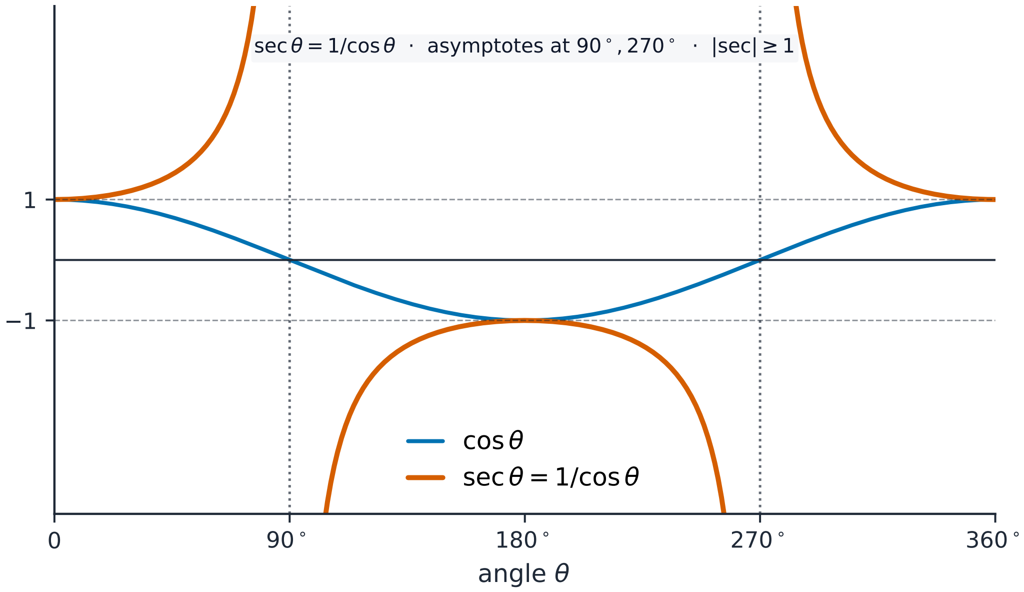
Build the skills to answer exam questions on **further trigonometry** 三角学 — the reciprocal functions, the identities $\sec^2\theta \equiv 1+\tan^2\theta$, **compound** 复合角 and **double angle** 二倍角 formulae, and the **R-formula** 辅助角公式.

You must be able to:

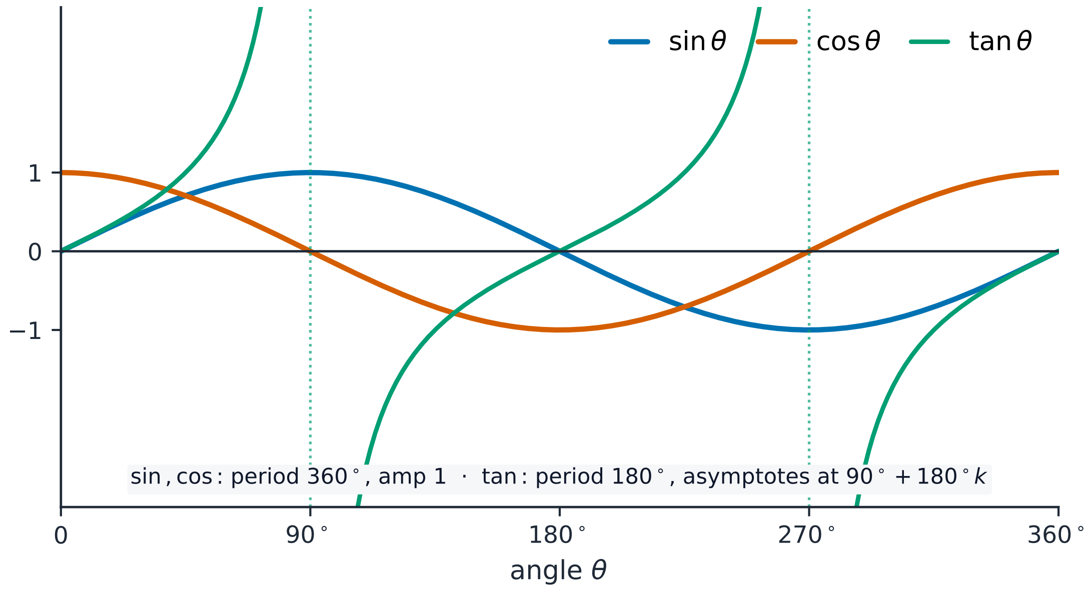
- use $\sec^2\theta \equiv 1+\tan^2\theta$ and $\csc^2\theta \equiv 1+\cot^2\theta$
- apply compound- and double-angle formulae
- write $a\sin\theta + b\cos\theta$ as $R\sin(\theta+\alpha)$ and solve

1 Worked examples

■ Reciprocal identity & R-formula



$\sec\theta = 1/\cos\theta$ *rises to an asymptote wherever* $\cos\theta = 0$



The underlying sine and cosine graphs

Solve $2\tan^2\theta + 3\sec\theta = 18$ **for** $-180° < \theta < 180°$**.** Use $\tan^2\theta = \sec^2\theta - 1$: $2\sec^2\theta + 3\sec\theta - 20 = 0 \Rightarrow (2\sec\theta - 5)(\sec\theta + 4) = 0$. $\cos\theta = \frac{2}{5}$ or $-\frac{1}{4} \Rightarrow \theta = \pm 66.4°,\ \pm 104.5°$.

R-formula: $a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)$, $R = \sqrt{a^2 + b^2}$, $\tan\alpha = \frac{b}{a}$.

2 Practice

2.1 Write $\sec^2\theta$ in terms of $\tan\theta$. [1]

2.2 Use a double-angle formula to write $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$. [1]

3 Exam-style questions

3.1 Expressed as $R\sin(\theta+\alpha)$, the value of R for $3\sin\theta + 4\cos\theta$ is: [1]

- **A** 5
- **B** 7
- **C** $\sqrt{7}$
- **D** 12

3.2 Express $5\sin\theta - 12\cos\theta$ in the form $R\sin(\theta-\alpha)$ with $R > 0$ and $0° < \alpha < 90°$. [3]

3.3 (a) Show that $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} \equiv 2\sec^2\theta$. [3]

(b) Hence solve $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 8$ for $0° \leqslant \theta \leqslant 180°$. [3]

Solutions

2.1 $\sec^2\theta = 1 + \tan^2\theta$.

2.2 $\sin 2\theta = 2\sin\theta\cos\theta$.

3.1 **A**. $R = \sqrt{3^2 + 4^2} = 5$.

3.2 $R = \sqrt{5^2 + 12^2} = 13$; $\tan\alpha = \frac{12}{5} \Rightarrow \alpha = 67.4°$; $13\sin(\theta - 67.4°)$.

3.3 (a) common denominator: $\dfrac{(1+\sin\theta)+(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} = \dfrac{2}{1-\sin^2\theta} = \dfrac{2}{\cos^2\theta} = 2\sec^2\theta$.
(b) $2\sec^2\theta = 8 \Rightarrow \sec^2\theta = 4 \Rightarrow \cos^2\theta = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$; $\theta = 60°$ or $120°$.

3 **(a)** Solve the equation $|2x-3|=|5x+2|$. [3]

(b) Hence solve the equation $|2\sec\theta-3|=|5\sec\theta+2|$ for $\pi < \theta < 2\pi$. Give your answer correct to 3 significant figures. [3]

4 Solve the equation $\cot\theta\tan(\theta+45°) = 7$ for $0° < \theta < 90°$. [5]

2 Solve the equation $2\tan^2\theta + 3\sec\theta = 18$ for $-180° < \theta < 180°$. [5]

7 **(a)** Prove that $\sin^2 2x + 4\cos^2 x \cos 2x \equiv 4\cos^4 x$. [3]

(b) Find the set of possible values of the constant k for which the equation

$$\sin^2 2x + 4\cos^2 x \cos 2x + 5 = k$$

has no real solutions. [2]

(c) Find the exact value of $\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \sqrt{\sin^2 t + 4\cos^2\left(\frac{1}{2}t\right)\cos t}\,\mathrm{d}t$. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

7 **(a)** Express $4\cos\theta\sin(\theta+30°)$ in the form $R\cos(2\theta-\alpha)+k$, where $R>0$, $0°<\alpha<90°$ and k is a constant. [6]

(b) Hence solve the equation

$$12\cos 2\phi \sin(2\phi + 30°) = 5$$

for $0° < \phi < 90°$. [5]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

2.4 Differentiation (product, quotient, parametric, implicit)

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS parametric 参数 • parametric equations 参数方程 • product 乘积法则 • implicit 隐式 • quotient rule 商法则

Objective

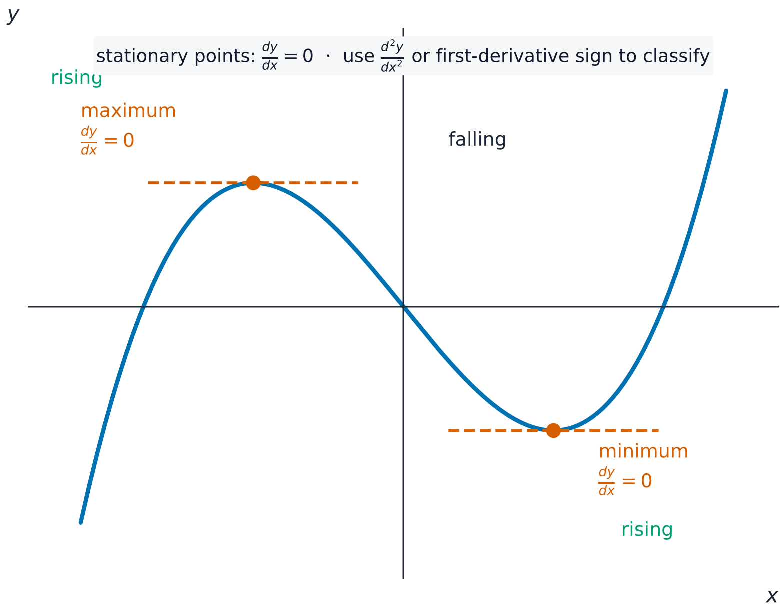
Build the skills to answer exam questions on **differentiation 微分** — standard derivatives (e^x, $\ln x$, trig), the **product 乘积法则** and **quotient rules 商法则**, and **parametric 参数** and **implicit 隐式** differentiation.

You must be able to:

- differentiate e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$
- apply the product and quotient rules
- differentiate parametric and implicit relations

1 Worked examples

■ Product rule & implicit



A stationary point ($\frac{dy}{dx} = 0$) is still found by differentiating — now with the product/quotient/chain rules

$y = 6x\cos(x^2 + 1)$. Product rule ($u = 6x$, $v = \cos(x^2 + 1)$):

$$\frac{dy}{dx} = 6\cos(x^2 + 1) - 12x^2 \sin(x^2 + 1).$$

Implicit: for $x^2 + y^2 = 25$, differentiate: $2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$.

2 Practice

2.1 Differentiate $y = x^2 e^x$. [2]

2.2 Differentiate $y = \dfrac{\sin x}{x}$. [2]

3 Exam-style questions

3.1 $\dfrac{d}{dx}(\ln(3x))$ equals: [1]

- **A** $\dfrac{1}{3x}$
- **B** $\dfrac{3}{x}$
- **C** $\dfrac{1}{x}$
- **D** $3\ln x$

3.2 A curve has parametric equations $x = t^2$, $y = t^3 - 3t$. Find $\dfrac{dy}{dx}$ in terms of t, and the value of t at the stationary point(s). [4]

3.3 A curve is defined implicitly by $x^2 + xy + y^2 = 7$.

(a) Find $\dfrac{dy}{dx}$ in terms of x and y. [3]

(b) Find the gradient of the curve at the point $(1, 2)$. [2]

Solutions

2.1 $\dfrac{dy}{dx} = 2xe^x + x^2e^x = xe^x(2+x)$.

2.2 $\dfrac{dy}{dx} = \dfrac{x\cos x - \sin x}{x^2}$.

3.1 **C**. $\ln(3x) = \ln 3 + \ln x$, so the derivative is $\dfrac{1}{x}$.

3.2 $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2 - 3$; $\dfrac{dy}{dx} = \dfrac{3t^2-3}{2t}$; stationary where $3t^2 - 3 = 0 \Rightarrow t = \pm 1$.

3.3 (a) differentiate: $2x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$.
(b) at $(1,2)$: $\dfrac{dy}{dx} = -\dfrac{2+2}{1+4} = -\dfrac{4}{5}$.

6 A curve has parametric equations

$$x = \tan\theta, \quad y = \sin\theta - 2\sin^3\theta,$$

for $0 < \theta < \frac{1}{2}\pi$.

(a) Show that $\dfrac{dy}{dx} = 6\cos^5\theta - 5\cos^3\theta$. [4]

(b) Find the equation of the normal to the curve at the point where it crosses the x-axis. Give your answer in the form $y = mx + c$, where m and c are exact constants. [5]

8 The equation of a curve is $y = 4e^{1-2x}\sqrt{3x-1}$.

Find the exact coordinates of the stationary point of the curve. [6]

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Additional page

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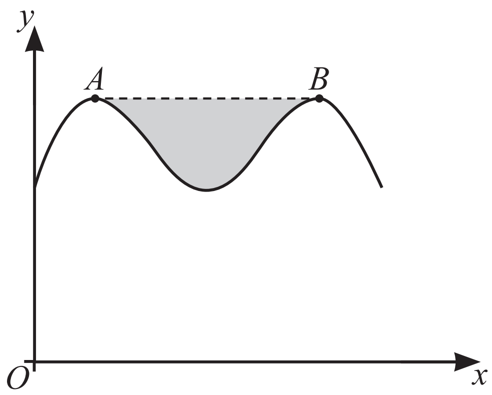

The diagram shows the curve with equation $y = 4\cos 2x + 8\sin x$ for $0 \leqslant x \leqslant \pi$. The maximum points on the curve are denoted by A and B, and the shaded region is bounded by the line segment AB and the curve.

(a) Find the coordinates of A and B. [5]

(b) Find the exact area of the shaded region. [6]

7 A curve has equation $5x^2y+4e^{2y}-7x+10=0$.

(a) Find an expression in terms of x and y for $\frac{dy}{dx}$ and hence find the gradient of the curve at the point for which $y=0$. [6]

(b) Show that there is no point on the curve at which the tangent is parallel to the y-axis. [2]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

1 Given that $y = 6x\cos(x^2 + 1)$, find an expression for $\frac{dy}{dx}$. [2]

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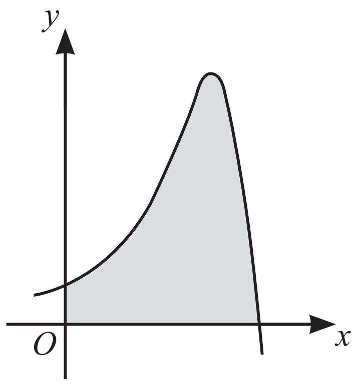

The diagram shows the curve with equation $y = 6e^{2x} - e^{3x}$. The shaded region is bounded by the axes and the curve.

(a) Find the exact x-coordinate of the maximum point. [3]

(b) Find the area of the shaded region. Give your answer in the form $\frac{p}{q}$, where p and q are integers. [4]

6 The parametric equations of a curve are

$$x = \frac{2t+1}{3t+4}, \qquad y = 2\ln(3t+4),$$

where $t > -\frac{4}{3}$.

(a) Show that $\frac{dy}{dx}$ can be expressed in the form $c(3t+4)$ and state the value of the constant c. [5]

(b) It is given that the gradient of the curve at the point $(a, \ln 100)$ is m.

Find the values of a and m. [4]

(c) State whether the curve represents a decreasing function or an increasing function or neither. Give a reason for your answer. [1]

3 Find the coordinates of the stationary points of the curve with equation $y = \frac{6x}{2x+3} - 6x + 5$. [5]

6 A curve has equation $(x^2 - 3)\ln y + 6x = 14$.

(a) Show that there is no point on the curve at which the y-coordinate is e^{-1}. [3]

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(b) Find the equation of the tangent to the curve at the point $(2, e^2)$. Give your answer in the form $y = mx + c$, where m and c are exact constants. [6]

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2.5 Integration (standard forms, trapezium rule)

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

KEY WORDS trapezium rule 梯形法则 • integration 积分

Objective

Build the skills to answer exam questions on **integration 积分** — integrating e^{ax+b}, $\dfrac{1}{ax+b}$ and trig functions, powers of sin/cos via identities, and the **trapezium rule 梯形法则**.

You must be able to:

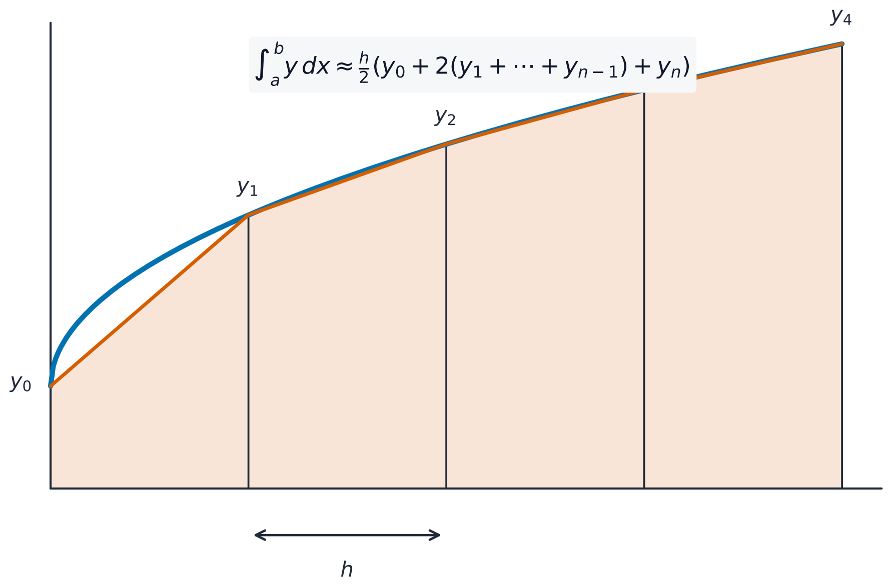
- integrate e^{ax+b}, $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$, $\sec^2(ax+b)$
- integrate $\sin^2/\cos^2$ using a double-angle identity
- estimate a definite integral with the trapezium rule

1 Worked examples

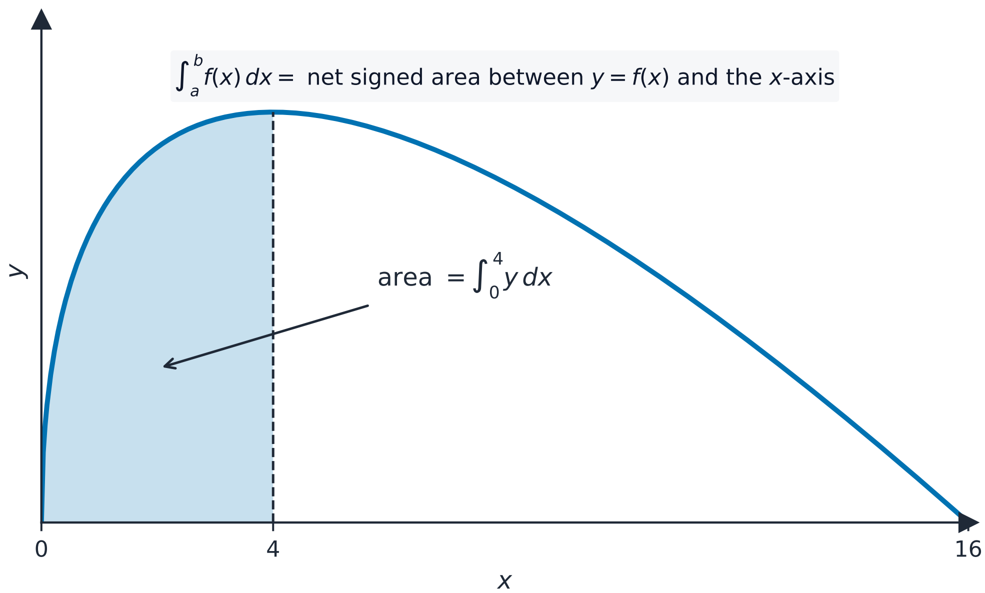
Powers of sin & the trapezium rule

$\displaystyle\int 6\sin^2 x\,dx$ — use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$:

$$\int (3 - 3\cos 2x)\,dx = 3x - \tfrac{3}{2}\sin 2x + C.$$



Each strip is a trapezium; their areas estimate
$\int_a^b y\,dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})]$



The trapezium rule estimates this area under the curve

2 Practice

2.1 Find $\int e^{2x+1}\,dx$. [2]

2.2 Find $\int \frac{1}{2x+3}\,dx$. [2]

3 Exam-style questions

3.1 $\int \cos 4x\,dx$ equals: [1]

- **A** $4\sin 4x + C$
- **B** $\frac{1}{4}\sin 4x + C$
- **C** $-\frac{1}{4}\sin 4x + C$
- **D** $-4\sin 4x + C$

3.2 Find $\int_0^{\pi/4} 4\cos^2 x\,dx$, giving an exact answer. [4]

3.3 Use the trapezium rule with **four** strips to estimate $\int_0^2 \sqrt{1+x^3}\,dx$, giving your answer to 3 s.f. [4]

Solutions

2.1 $\frac{1}{2}e^{2x+1} + C$.

2.2 $\frac{1}{2}\ln|2x+3| + C$.

3.1 **B**. $\int \cos 4x\, dx = \frac{1}{4}\sin 4x + C$.

3.2 $4\cos^2 x = 2(1+\cos 2x)$; $\displaystyle\int_0^{\pi/4}(2+2\cos 2x)\, dx = [2x+\sin 2x]_0^{\pi/4}$; $= \left(\frac{\pi}{2}+1\right) - 0 = \frac{\pi}{2}+1$.

3.3 $h = 0.5$; y at $x = 0, 0.5, 1, 1.5, 2$: 1, $\sqrt{1.125} = 1.0607$, $\sqrt{2} = 1.4142$, $\sqrt{4.375} = 2.0917$, $\sqrt{9} = 3$; $\int \approx \frac{0.5}{2}[1+3+2(1.0607+1.4142+2.0917)]$; $= 0.25[4+2(4.5666)] = 0.25(13.133) = 3.28$ (3 s.f.).

1 Find $\int 6\sin^2 x \, dx$. [3]

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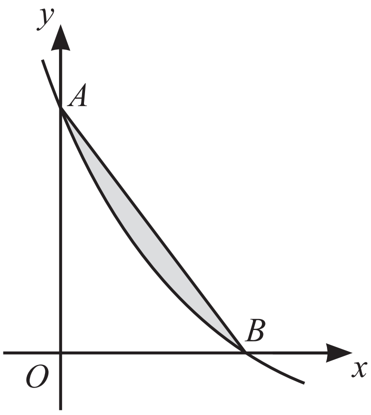

The diagram shows the curve with equation $y = 8e^{-\frac{1}{2}x} - 1$. The curve meets the axes at the points A and B. The shaded region is bounded by the curve and the line segment AB.

(a) Show that the x-coordinate of B is $6\ln 2$. [2]

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(b) Find the area of the shaded region. Give your answer in the form $p\ln 2 - q$, where p and q are positive integers. [5]

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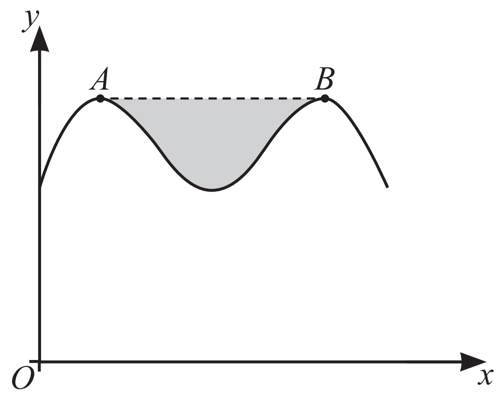

The diagram shows the curve with equation $y = 4\cos 2x + 8\sin x$ for $0 \leqslant x \leqslant \pi$. The maximum points on the curve are denoted by A and B, and the shaded region is bounded by the line segment AB and the curve.

(a) Find the coordinates of A and B. [5]

(b) Find the exact area of the shaded region. [6]

6 **(a)** Given that $\int_{-2a}^{a}\left(\frac{1}{2}e^{2x}+\frac{1}{4}e^{-x}\right)dx = 5$, where a is a positive constant, show that

$$a = \frac{1}{2}\ln\left(10+\frac{1}{2}e^{-a}+\frac{1}{2}e^{-4a}\right).$$

[4]

(b) Hence show by calculation that the value of a lies between 1.0 and 1.2. [2]

(c) Use the iterative formula

$$a_{n+1} = \frac{1}{2}\ln\left(10 + \frac{1}{2}e^{-a_n} + \frac{1}{2}e^{-4a_n}\right)$$

to find the value of a correct to 4 significant figures. Give the result of each iteration to 6 significant figures. [3]

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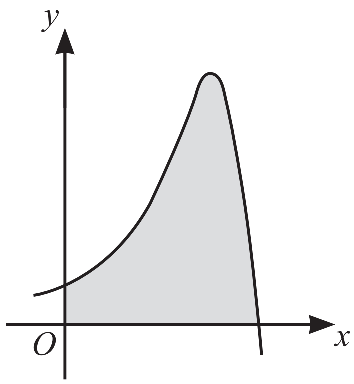

The diagram shows the curve with equation $y = 6e^{2x} - e^{3x}$. The shaded region is bounded by the axes and the curve.

(a) Find the exact x-coordinate of the maximum point. [3]

(b) Find the area of the shaded region. Give your answer in the form $\dfrac{p}{q}$, where p and q are integers. [4]

7 **(a)** Prove that $\sin^2 2x + 4\cos^2 x \cos 2x \equiv 4\cos^4 x$. [3]

(b) Find the set of possible values of the constant k for which the equation

$$\sin^2 2x + 4\cos^2 x \cos 2x + 5 = k$$

has no real solutions. [2]

(c) Find the exact value of $\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \sqrt{\sin^2 t + 4\cos^2\left(\frac{1}{2}t\right)\cos t}\,\mathrm{d}t$. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

1 Show that $\int_2 \frac{8}{4x+1}\,dx = \ln a$, where a is an integer to be found. [3]

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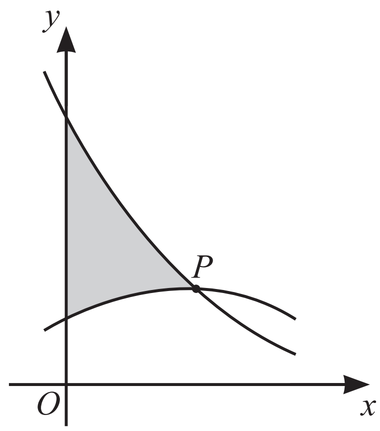


The diagram shows parts of the curves with equations $y = 4e^{-2x}$ and $y = 1 + 0.5\sin 3x$. Point P is a point of intersection of the curves, and the shaded region is bounded by the two curves and the y-axis.

(a) Show that the x-coordinate of P satisfies the equation $x = -0.5\ln(0.25 + 0.125\sin 3x)$. [1]

(b) Use an iterative formula, based on the equation in part **(a)**, to find the x-coordinate of P correct to 4 significant figures. Use an initial value of 0.5 and give the result of each iteration to 6 significant figures. [3]

(c) Hence find the area of the shaded region. Give your answer correct to 2 significant figures. [4]

2.6 Numerical solution of equations

Name: ______________ Class: _________ Date: ____________ **Total: 12 marks**

KEY WORDS iterative formula 迭代公式 • sign change 变号 • iteration 迭代 • root 根 • converge 收敛

Objective

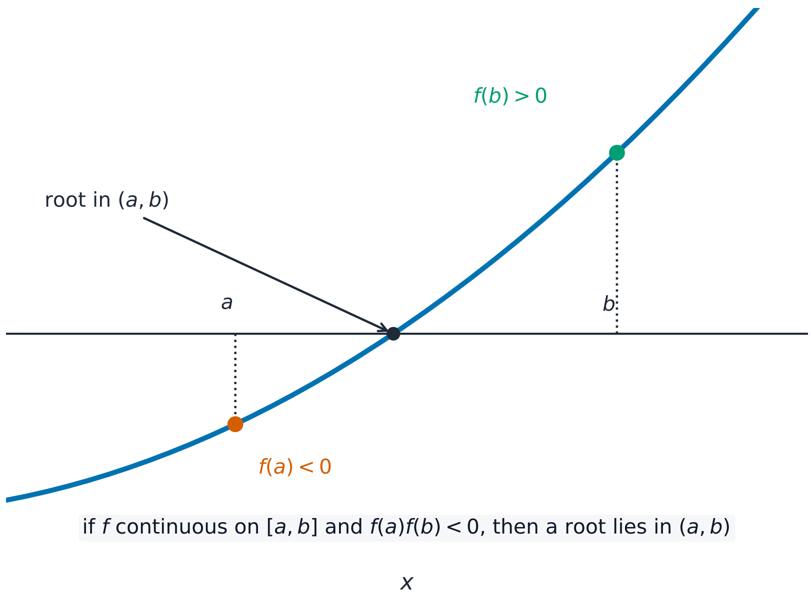
Build the skills to answer exam questions on the **numerical solution of equations** 方程的数值解 — locating a **root** 根 by a **sign change** 变号, and finding it by **iteration** 迭代.

You must be able to:

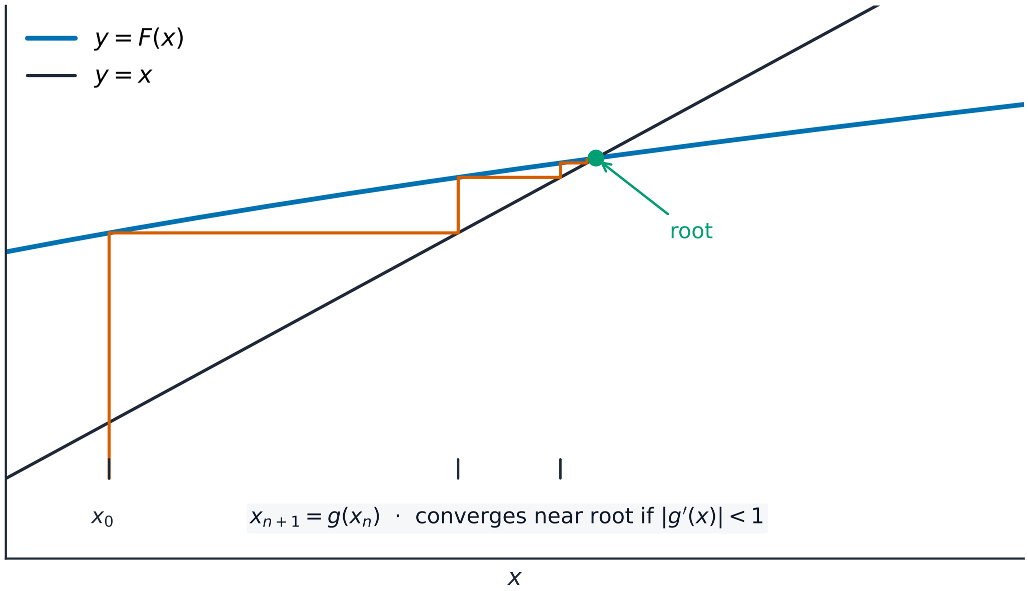
- use a sign change to show a root lies in an interval
- carry out an iteration $x_{n+1} = F(x_n)$ until it converges
- show a given equation rearranges into a stated iterative form

1 Worked examples

■ Sign change & iteration



If $f(a)$ and $f(b)$ have opposite signs, a root is trapped between a and b



Each step: up to $y = F(x)$, across to $y = x$; the staircase converges to the root

Iterate $x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ **from** $x_0 = -1.2$**:** $x_1 = -1.281$, $x_2 = -1.247$, ... $\rightarrow \beta = -1.26$ (3 s.f.).

2 Practice

2.1 $f(x) = x^3 - 5x + 1$. Evaluate $f(0)$ and $f(1)$, and state what they tell you. [2]

2.2 State the iterative formula you would use, written as $x_{n+1} = F(x_n)$, for the equation $x = \cos x$. [1]

3 Exam-style questions

3.1 A root of $f(x) = 0$ lies between $x = 2$ and $x = 3$ if: [1]

- **A** $f(2)$ and $f(3)$ are both positive
- **B** $f(2)$ and $f(3)$ have opposite signs
- **C** $f(2) = f(3)$
- **D** $f'(2) = 0$

3.2 The equation $x^3 - 6x + 2 = 0$ has a root α between 0 and 1.

(a) Verify this by a sign change. [2]

(b) Show that the equation can be written as $x = \dfrac{x^3 + 2}{6}$, and use the iteration $x_{n+1} = \dfrac{x_n^3 + 2}{6}$ with $x_0 = 0.3$ to find α to 2 d.p. [4]

3.3 Explain why a sign change between a and b does **not** guarantee exactly one root in that interval. [2]

Solutions

2.1 $f(0) = 1,\ f(1) = 1 - 5 + 1 = -3$; opposite signs, so a root lies between 0 and 1.

2.2 $x_{n+1} = \cos x_n$.

3.1 **B**. Opposite signs (with a continuous graph) trap a root between them.

3.2 (a) $f(0) = 2 > 0,\ f(1) = 1-6+2 = -3 < 0$; opposite signs $\to$ root between 0 and 1.
(b) $x^3 - 6x + 2 = 0 \Rightarrow 6x = x^3 + 2 \Rightarrow x = \dfrac{x^3+2}{6}$; $x_1 = \dfrac{0.027+2}{6} = 0.3378$, $x_2 = 0.3397$, $x_3 = 0.3398$; $\alpha = 0.34$.

3.3 there could be several roots between a and b — the graph may cross the axis more than once (an odd number of times).

7 The polynomial $p(x)$ is defined by

$$p(x) = 2x^4 + kx^3 + kx^2 + 17x + 18,$$

where k is a constant. It is given that $(x+2)$ is a factor of $p(x)$.

(a) Find the value of k. [2]

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It is given that the equation $p(x) = 0$ has exactly two real roots, denoted by α and β, where α is an integer and β is not an integer.

(b) State the value of α and show that β satisfies the equation $x = \sqrt[3]{-2x-4.5}$. [4]

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(c) Show by calculation that $-1.4 < \beta < -1.0$. [2]

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(d) Use an iterative formula, based on the equation in part **(b)**, to find the value of β correct to 3 significant figures. Give the result of each iteration to 5 significant figures. [3]

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6 **(a)** Given that $\int_{-2a} \left(\frac{1}{2}e^{2x} + \frac{1}{4}e^{-x}\right) dx = 5$, where a is a positive constant, show that

$$a = \frac{1}{2}\ln\left(10 + \frac{1}{2}e^{-a} + \frac{1}{2}e^{-4a}\right).$$ [4]

(b) Hence show by calculation that the value of a lies between 1.0 and 1.2. [2]

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(c) Use the iterative formula

$$a_{n+1} = \frac{1}{2}\ln\left(10 + \frac{1}{2}e^{-a_n} + \frac{1}{2}e^{-4a_n}\right)$$

to find the value of a correct to 4 significant figures. Give the result of each iteration to 6 significant figures. [3]

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3 **(a)** Sketch, on a single diagram, the graphs of $y = 3e^{-2x}$ and $y = \sec x$ for values of x such that $0 \leqslant x < \frac{1}{2}\pi$. [2]

(b) Show that the x-coordinate of the point of intersection of the two graphs satisfies the equation $x = \frac{1}{2}\ln(3\cos x)$. [2]

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(c) Use an iterative formula, based on the equation in part **(b)**, to find the x-coordinate of the point of intersection correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

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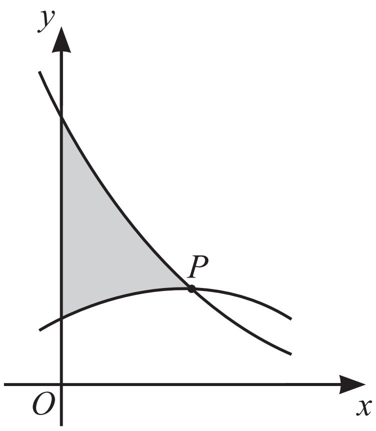

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(c) Hence find the area of the shaded region. Give your answer correct to 2 significant figures. [4]