Week 1

A-Level Mathematics

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

A-Level Mathematics Name: ____________ Score: ______

1. For $x^2 + 4x + 1 = 0$, what is the discriminant $b^2 - 4ac$?

 ____

2. If $f(x) = x^2$ and $g(x) = x + 1$, what is $fg(2) = f(g(2))$?

 ____

3. What is the gradient of the line through (2, 3) and (6, 11)?

 ____

4. π radians equals how many degrees?

 ____

5. What is the exact value of sin 30°?

 ____

6. The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line:
   - ☐ y = x
   - ☐ the x-axis
   - ☐ the y-axis
   - ☐ y = -x
7. At a stationary point, if the second derivative $f''(x) > 0$, the point is a:
   - ☐ minimum
   - ☐ maximum
   - ☐ point of inflection
   - ☐ root
8. Why does an indefinite integral include ”+ C”?
   - ☐ many curves share the same derivative, differing by a constant
   - ☐ C is always zero
   - ☐ it marks the limits
   - ☐ it is the area
9. If $b^2 - 4ac < 0$, the quadratic equation has:
   - ☐ no real roots

- ☐ one repeated root
- ☐ two distinct roots
- ☐ three roots

10. Only a one-one function has an inverse function.

☐ True ☐ False

A-Level Mathematics

1. **12**

   $b^2 - 4ac = 4^2 - 4(1)(1) = 16 - 4 = 12$ (> 0, so two distinct real roots).

2. **9**

   $g(2) = 3$, then $f(3) = 3^2 = 9$.

3. **2**

   $m = (11 - 3) / (6 - 2) = 8/4 = 2$.

4. **180**

   By definition, π radians = 180°.

5. **0.5**

   sin 30° = 1/2 = 0.5.

6. **y = x**

   An inverse reflects the original graph in the line $y = x$; a point (a, b) becomes (b, a).

7. **minimum**

   $f'' > 0$ means the curve bends upward — a minimum; $f'' < 0$ gives a maximum.

8. **many curves share the same derivative, differing by a constant**

   Differentiating a constant gives 0, so the original could have had any constant — hence + C.

9. **no real roots**

   A negative discriminant means the parabola never crosses the x-axis — no real roots.

10. **True**

    If two inputs gave the same output, the reverse rule would be ambiguous — so only one-one functions are invertible.

1.1 Quadratics

Name: ____________ Class: ________ Date: __________ **Total: 16 marks**

KEY WORDS quadratic 二次式 • quadratic equation 二次方程 • discriminant 判别式 • simultaneous equations 联立 • real roots 实根

Objective

Build the skills to answer exam questions on **quadratics** 二次式 — completing the square, the **discriminant** 判别式, solving equations and inequalities, and simultaneous equations.

You must be able to:

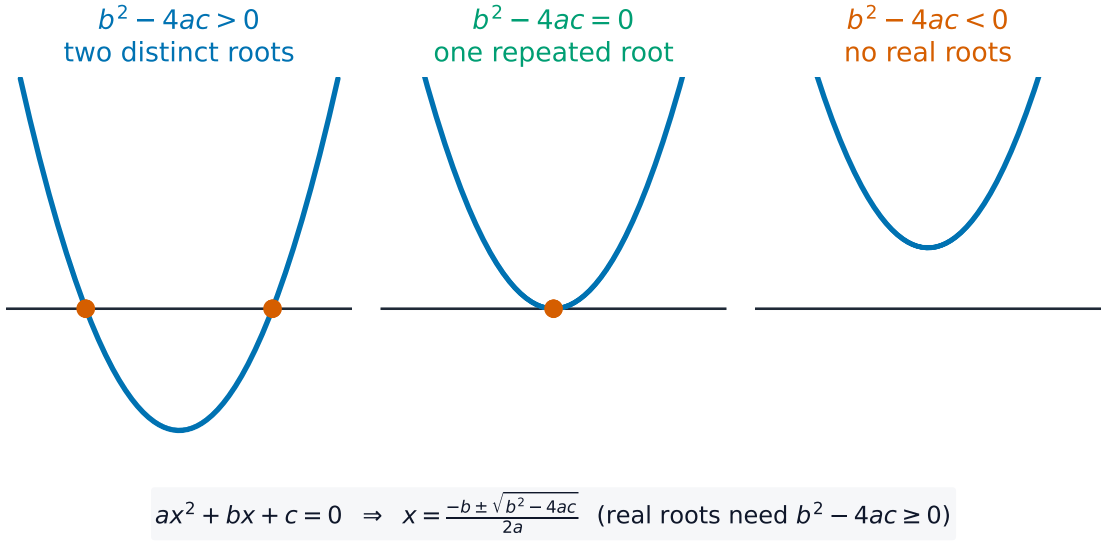
- write $ax^2 + bx + c$ in completed-square form $a(x+p)^2 + q$
- use $b^2 - 4ac$ to decide the number of real roots
- solve quadratic equations and inequalities
- solve a linear–quadratic pair by substitution

1 Worked examples

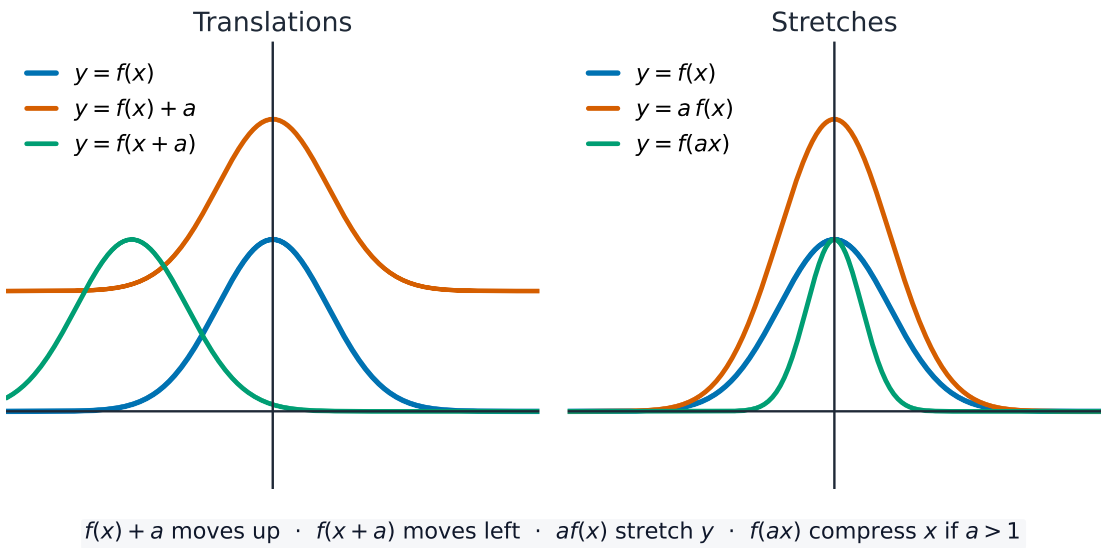
Completing the square & the discriminant

Write $2x^2 - 12x + 5$ **as** $a(x+p)^2 + q$**.**

$$2x^2 - 12x + 5 = 2(x^2 - 6x) + 5 = 2((x-3)^2 - 9) + 5 = 2(x-3)^2 - 13.$$



The sign of $b^2 - 4ac$ decides how many times the parabola meets the x-axis



Completing the square shifts and stretches the parabola

Find the values of k for which $x^2 + kx + 9 = 0$ has two distinct real roots.

Two distinct roots need $b^2 - 4ac > 0$: $k^2 - 36 > 0 \Rightarrow (k-6)(k+6) > 0 \Rightarrow k < -6$ or $k > 6$.

2 Practice

2.1 Express $x^2 + 8x + 3$ in the form $(x + p)^2 + q$. [2]

2.2 State the condition on the discriminant for a quadratic to have **no** real roots. [1]

3 Exam-style questions

3.1 The equation $2x^2 - 4x + c = 0$ has a repeated root. The value of c is: [1]

- **A** 0
- **B** 1
- **C** 2
- **D** 4

3.2 Express $3x^2 + 12x + 7$ in the form $a(x + b)^2 + c$ and hence write down the least value of the expression. [4]

3.3 Find the set of values of k for which the line $y = kx - 2$ meets the curve $y = x^2 - 3x + 2$ at two distinct points. [4]

3.4 Solve the simultaneous equations $y = 2x - 1$ and $x^2 + y^2 = 2$. [4]

Solutions

2.1 $(x+4)^2 - 13$.

2.2 $b^2 - 4ac < 0$.

3.1 **C**. Repeated root needs $b^2 - 4ac = 0$: $16 - 8c = 0 \Rightarrow c = 2$.

3.2 $3(x^2+4x)+7 = 3((x+2)^2-4)+7 = 3(x+2)^2-5$; least value $= -5$ (when $x = -2$).

3.3 set equal: $x^2 - 3x + 2 = kx - 2 \Rightarrow x^2 - (3+k)x + 4 = 0$; two points need $b^2 - 4ac > 0$: $(3+k)^2 - 16 > 0$; $(k+7)(k-1) > 0$; $k < -7$ or $k > 1$.

3.4 substitute: $x^2 + (2x-1)^2 = 2 \Rightarrow 5x^2 - 4x - 1 = 0$; $(5x+1)(x-1) = 0 \Rightarrow x = -\frac{1}{5}$ or $x = 1$; $y = 2x - 1$ gives $(1, 1)$ and $\left(-\frac{1}{5}, -\frac{7}{5}\right)$.

1 Find the set of values of the constant k for which the quadratic equation

$$3kx^2 + (k+8)x + 3 = 0$$

has two distinct real roots. [4]

4 **(a)** Express $1-6x-x^2$ in the form $a-(x+b)^2$, where a and b are constants. [3]

(b) The graph of $y=x^2$ is transformed to the graph of $y=1-6x-x^2$ by a reflection followed by a translation of $\begin{pmatrix} m \\ n \end{pmatrix}$. Give details of the reflection and determine the values of m and n. [3]

1 **(a)** Express $9x^2 - 36x + 8$ in the form $p(x+q)^2 + r$, where p, q and r are constants. [2]

(b) Hence find the set of values of the constant k for which the equation $9x^2 - 36x + 8 = k$ has no real roots. [1]

(c) Find the exact roots of the equation $9x^2 - 36x + 8 = -15$. [2]

2 Find the coordinates of the points of intersection of the curve and the line with equations

$$2xy + 5y^2 = 24 \quad \text{and} \quad 2x + y + 4 = 0.$$

[4]

11 **(a)** Express $x^2 + 4x + 2$ in the form $(x+a)^2 + b$, where a and b are integers. [2]

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The functions f and g are defined as follows.

$$\mathrm{f}(x) = x^2 + 4x + 2 \qquad \text{for } x \leqslant -2$$

$$\mathrm{g}(x) = -x - 4 \qquad \text{for } x \geqslant -2$$

(b) **(i)** Find an expression for $\mathrm{f}^{-1}(x)$. [3]

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(ii) Find an expression for $(\mathrm{gf})^{-1}(x)$. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

1.2 Functions

Name: ____________ Class: ________ Date: __________ **Total: 12 marks**

KEY WORDS functions 函数 • transformations 变换 • domain 定义域 • composition 复合 • inverse function 反函数

Objective

Build the skills to answer exam questions on **functions** 函数 — domain and range, **composition** 复合, **inverse functions** 反函数, and **transformations** 变换 of graphs.

You must be able to:

- find the range of a function and a composite fg
- find an inverse f^{-1} and state its domain
- describe and apply graph transformations

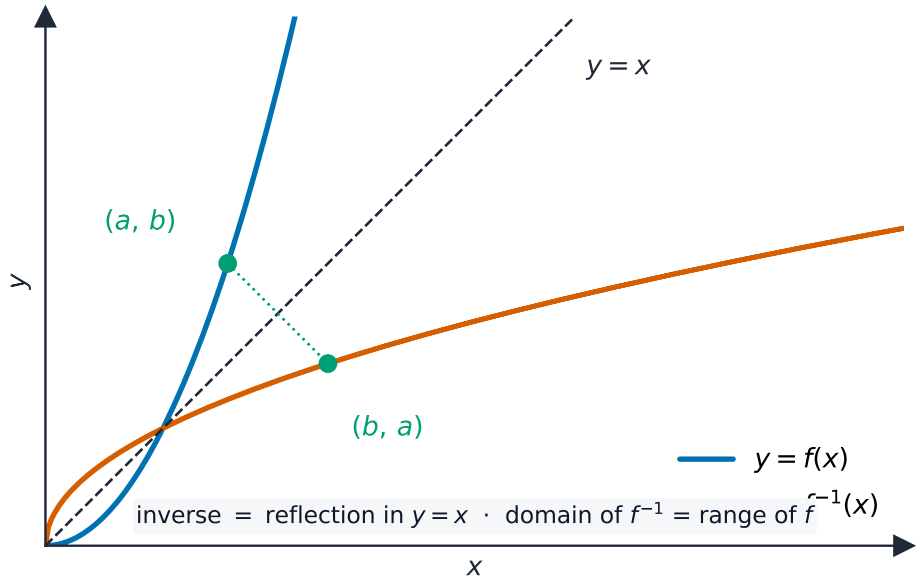
1 Worked examples

Inverse & composition

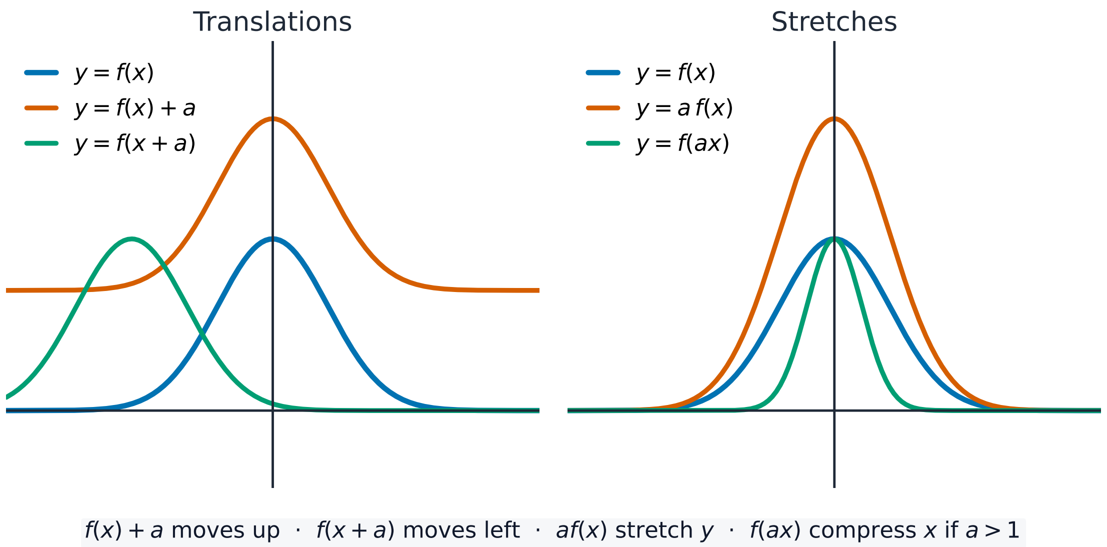
$f(x) = (x+3)^2 - 12$ for $x \geqslant 0$. Find $f^{-1}(x)$.

$$y = (x+3)^2 - 12 \Rightarrow (x+3)^2 = y + 12 \Rightarrow x = \sqrt{y+12} - 3.$$

So $f^{-1}(x) = \sqrt{x+12} - 3$ (positive root, since $x \geqslant 0$).



The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$



$y = f(x) + a$ slides it up; $y = f(x + a)$ slides it left; $y = a\,f(x)$ stretches it in the y-direction

2 Practice

2.1 $f(x) = 2x - 5$ and $g(x) = x^2$. Find $fg(3)$. [2]

2.2 Describe the single transformation mapping $y = f(x)$ to $y = f(x) - 4$. [1]

3 Exam-style questions

3.1 $f(x) = 3 - 2x$. Then $f^{-1}(x)$ is: [1]

- **A** $\dfrac{3-x}{2}$
- **B** $\dfrac{x-3}{2}$
- **C** $2x - 3$
- **D** $\dfrac{3+x}{2}$

3.2 The function f is defined by $f(x) = x^2 - 6x + 5$ for $x \geqslant 3$.

(a) Express $f(x)$ in completed-square form and state the range of f. [3]

(b) Find $f^{-1}(x)$ and state its domain. [3]

3.3 The graph of $y = f(x)$ is transformed to $y = 2f(x + 1)$. Describe the **two** transformations, in the correct order. [2]

Solutions

2.1 $g(3) = 9$; $f(9) = 2(9) - 5 = 13$.

2.2 a translation **down** by 4.

3.1 **A**. $y = 3 - 2x \Rightarrow x = \dfrac{3-y}{2}$, so $f^{-1}(x) = \dfrac{3-x}{2}$.

3.2 (a) $x^2 - 6x + 5 = (x-3)^2 - 4$; for $x \geqslant 3$, range is $f(x) \geqslant -4$.
(b) $y = (x-3)^2 - 4 \Rightarrow x = 3 + \sqrt{x+4} \to f^{-1}(x) = 3 + \sqrt{x+4}$; domain $x \geqslant -4$.

3.3 translation left by 1 ($x \to x+1$); then a stretch in the y-direction, scale factor 2.

4 **(a)** Express $1-6x-x^2$ in the form $a-(x+b)^2$, where a and b are constants. [3]

(b) The graph of $y=x^2$ is transformed to the graph of $y=1-6x-x^2$ by a reflection followed by a translation of $\begin{pmatrix} m \\ n \end{pmatrix}$. Give details of the reflection and determine the values of m and n. [3]

6 Functions f and g are defined by

$$f(x) = (x+3)^2 - 12 \quad \text{for } x \geqslant 0,$$

$$g(x) = 2x - 5 \quad \text{for } x \in \mathbb{R}.$$

(a) State the range of f. [1]

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(b) Find an expression for $f^{-1}(x)$. [2]

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(c) Solve the equation $gf(x) = 69$. [4]

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3 **(a)** The graph of $y = f(x)$ is transformed to the graph of $y = f(3x) + 2$.

Describe fully the two transformations which have been combined to give the resulting graph. [3]

(b) A different graph has equation $y = g(x)$. This graph is stretched by scale factor 3 in the y-direction and then reflected in the y-axis.

Write down the equation of the transformed graph in terms of the function g. [2]

9 The function f is defined by $\mathrm{f}(x) = \dfrac{4}{(3x-6)^2} + \dfrac{1}{(3x-6)^3}$ for $x > 2$.

(a) Find an expression for $\mathrm{f}'(x)$ and hence determine whether f is an increasing function, a decreasing function or neither. [4]

(b) State whether f^{-1} exists. Give a reason for your answer. [1]

The function g is defined by $g(x) = 4x - 3$ for $x > a$.

(c) Find the range of g in terms of the constant a. [1]

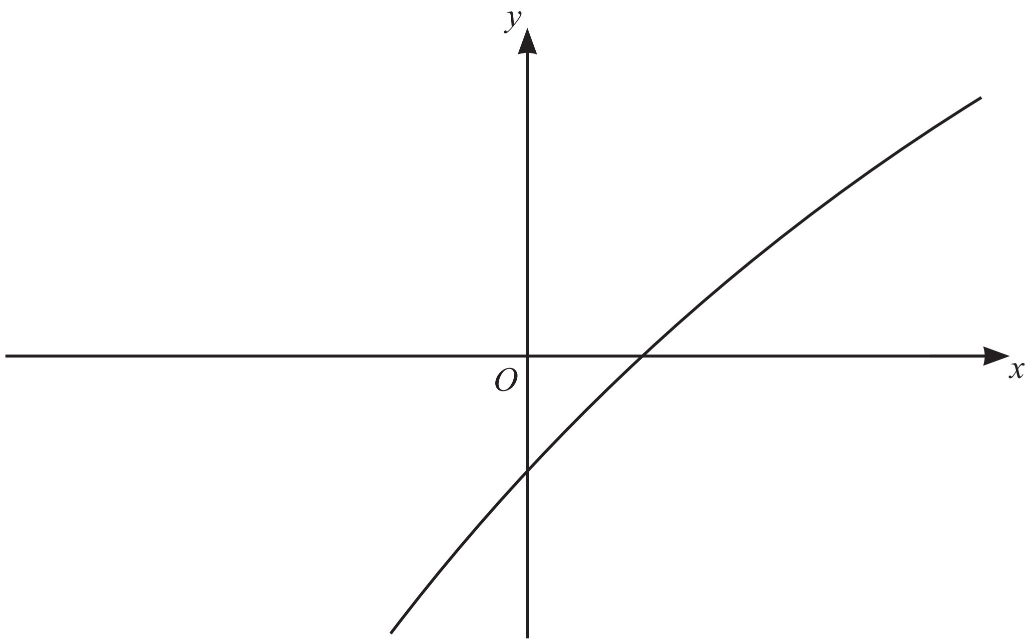
(d) Find the set of values of a for which the composite function fg exists. [2]

10 The functions f and g are defined by

$$\mathrm{f}(x) = \sqrt{x} \qquad \text{for } x \geqslant 0,$$

$$\mathrm{g}(x) = 3\sqrt{x+2} - 5 \qquad \text{for } x \geqslant -2.$$

(a) Describe fully a sequence of transformations which transforms the graph of $y = \mathrm{f}(x)$ to the graph of $y = \mathrm{g}(x)$. You should make clear the order in which the transformations are applied. [5]



The diagram shows the graph of $y = \mathrm{g}(x)$.

(b) On the diagram sketch the graph of $y = \mathrm{g}^{-1}(x)$ together with any relevant mirror line. [2]

(c) Find an expression for $g^{-1}(x)$. [2]

(d) State the range of g^{-1}. [1]

The function h is defined by

$$h(x) = x - 2 \qquad \text{for } x \geqslant 0.$$

(e) Find the value of $g^{-1}h(4)$. [1]

(f) Explain why the composite function hg^{-1} cannot be formed. [1]

Additional page

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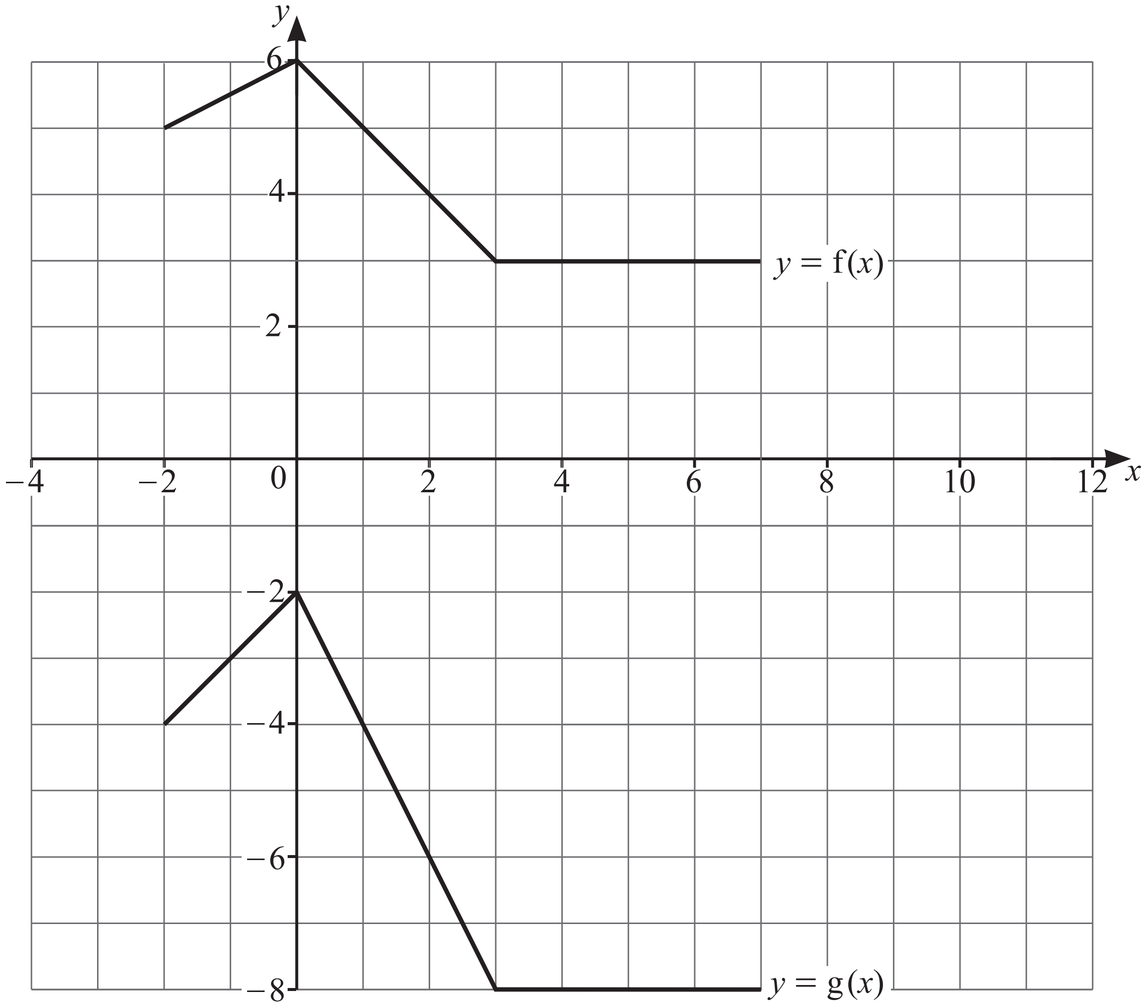

The diagram shows the graphs with equations $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$.

Describe fully a sequence of two transformations which transforms the graph of $y = \mathrm{f}(x)$ to the graph of $y = \mathrm{g}(x)$. Make clear the order in which the transformations should be applied. [4]

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11 **(a)** Express x^2+4x+2 in the form $(x+a)^2+b$, where a and b are integers. [2]

The functions f and g are defined as follows.

$$f(x)=x^2+4x+2 \qquad \text{for } x \leqslant -2$$

$$g(x)=-x-4 \qquad \text{for } x \geqslant -2$$

(b) **(i)** Find an expression for $f^{-1}(x)$. [3]

(ii) Find an expression for $(\mathrm{gf})^{-1}(x)$. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

1.3 Coordinate geometry

Name: ______________ Class: __________ Date: ____________ **Total: 16 marks**

KEY WORDS circle 圆 • tangent 切线 • straight line 直线 • gradient 斜率 • perpendicular 垂直

Objective

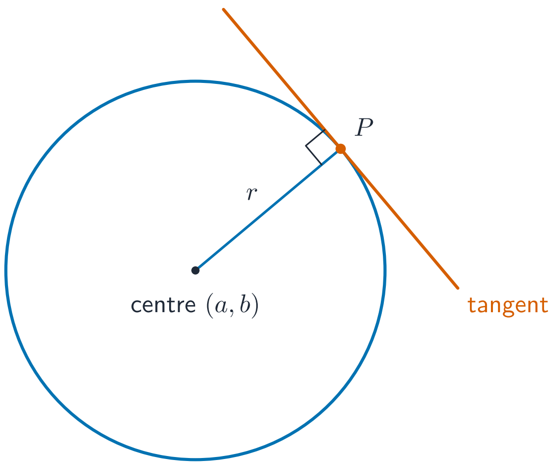
Build the skills to answer exam questions on **coordinate geometry** 坐标几何 — gradients, the equation of a **straight line** 直线, parallel/perpendicular lines, and the equation of a **circle** 圆.

You must be able to:

- find the gradient, midpoint and equation of a line
- use the perpendicular-gradient rule ($m_1 m_2 = -1$)
- find a circle's centre and radius; use the tangent–radius right angle

1 Worked examples

Circle and tangent



A tangent touches the circle once and meets the radius at a right angle

$P(1,1)$ and $Q(7,11)$ are ends of a diameter. Find the circle's equation.

Centre = midpoint = $(4,6)$; radius = $\frac{1}{2}\sqrt{6^2+10^2} = \frac{1}{2}\sqrt{136} = \sqrt{34}$. So $(x-4)^2 + (y-6)^2 = 34$.

Perpendicular: a line of gradient 2 has perpendicular gradient $-\frac{1}{2}$.

2 Practice

2.1 Find the gradient of the line joining $(2,-1)$ and $(6,7)$. [2]

2.2 Write the centre and radius of $(x-3)^2+(y+2)^2=25$. [2]

3 Exam-style questions

3.1 The line through $(0,4)$ perpendicular to $y=\frac{1}{3}x+1$ has gradient: [1]

- **A** $\frac{1}{3}$
- **B** 3
- **C** -3
- **D** $-\frac{1}{3}$

3.2 A circle has equation $x^2+y^2-6x+10y-2=0$.

(a) Find its centre and radius. [3]

(b) Show that the point $(3,1)$ lies on the circle. [2]

3.3 The points $A(1,2)$ and $B(5,10)$ are given.

(a) Find the equation of the perpendicular bisector of AB. [4]

(b) The perpendicular bisector meets the x-axis at C. Find the coordinates of C. [2]

Solutions

2.1 $m = \dfrac{7-(-1)}{6-2} = \dfrac{8}{4} = 2$.

2.2 centre $(3, -2)$; radius 5.

3.1 **C**. Perpendicular gradient $= -1 \div \frac{1}{3} = -3$.

3.2 (a) complete the square: $(x-3)^2 - 9 + (y+5)^2 - 25 - 2 = 0 \Rightarrow (x-3)^2 + (y+5)^2 = 36$; centre $(3, -5)$, radius 6.
(b) substitute $(3, 1)$: $(3-3)^2 + (1+5)^2 = 0 + 36 = 36 = 6^2$; this equals r^2, so $(3, 1)$ lies on the circle.

3.3 (a) midpoint $(3, 6)$; gradient $AB = \dfrac{10-2}{5-1} = 2$, so bisector gradient $= -\frac{1}{2}$; $y - 6 = -\frac{1}{2}(x-3) \Rightarrow y = -\frac{1}{2}x + \frac{15}{2}$.
(b) set $y = 0$: $0 = -\frac{1}{2}x + \frac{15}{2} \Rightarrow x = 15$, so $C(15, 0)$.

10 A circle has equation $x^2+y^2+4y-21=0$ and a straight line has equation $2x+y-8=0$.
The line intersects the circle at two points.

(a) Find the coordinates of these two points of intersection. [4]

(b) The circle has centre C and the two points of intersection are denoted by A and B.

Find the area of the triangle ABC. [3]

7 The coordinates of the points P and Q are (1, 1) and (7, 11) respectively. The line segment PQ forms a diameter of a circle.

(a) Find the equation of the circle. [4]

(b) Find the equation of the tangent to the circle at the point Q. [3]

(c) The other point on the circle with x-coordinate 7 is R.

Find the coordinates of the point of intersection of the tangent at Q with the tangent at R. [4]

6 The equation of a curve is $2x^2 - kxy + 2 = 0$ and the equation of a line is $y = px + 3$, where k and p are constants.

(a) Given that $k = 2$ and $p = 11$, find the coordinates of the points of intersection of the curve and the line. [4]

(b) Given instead that $p = 4$, find the set of values of k for which the curve and the line do not intersect. [5]

8 The circle with equation $x^2+y^2-6x+10y-27=0$ intersects the line $x=-2$ at the points P and Q.

Find the area of the triangle formed by the tangents to the circle at P and Q, and the line $x=-2$. [8]

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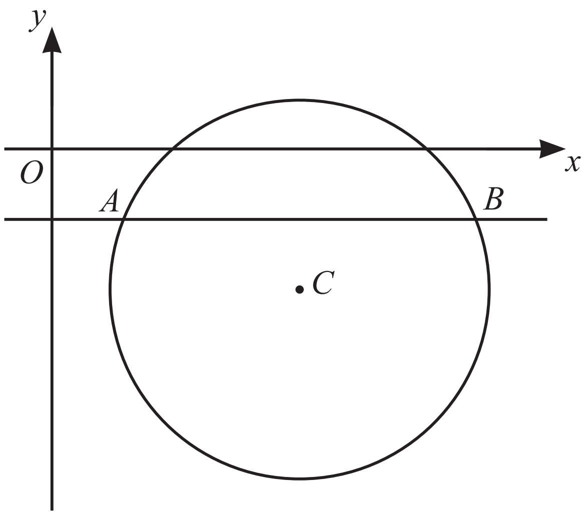


The diagram shows the circle with equation $x^2 + y^2 - 14x + 8y + 36 = 0$ and the line $y = -2$. The line intersects the circle at the points A and B. The centre of the circle is C.

(a) Find the coordinates of A, B and C. [3]

(b) Find the angle ACB in radians. Give your answer correct to 3 significant figures. [2]

(c) The chord AB divides the circle into two segments.

Find the area of the larger segment. [4]

A-LEVEL MATHEMATICS • EXERCISE SHEET

1.4 Circular measure

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

KEY WORDS radian 弧度 • sector 扇形 • arc length 弧长 • sector area 扇形面积 • segment 弓形

Objective

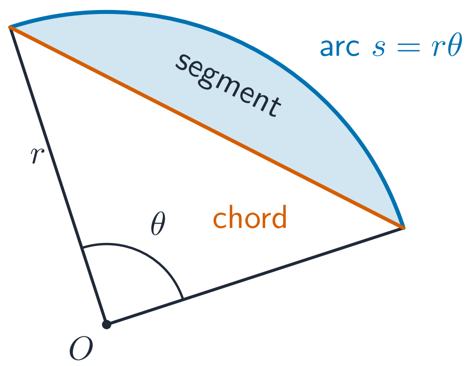
Build the skills to answer exam questions on **circular measure 弧度制** — **radians 弧度**, **arc length 弧长**, **sector area 扇形面积** and the area of a **segment 弓形**.

You must be able to:

- convert between degrees and radians
- use $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ (with θ in radians)
- find a segment area as sector minus triangle

1 Worked examples

■ Sector and segment



The shaded segment is the part of the sector between the chord and the arc

For a sector (angle θ in radians): $s = r\theta$, $A = \frac{1}{2}r^2\theta$. Segment $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$.

A sector has $r = 6$, $\theta = \frac{2}{3}\pi$. Arc $= 6 \cdot \frac{2}{3}\pi = 4\pi$; area $= \frac{1}{2}(36)\frac{2}{3}\pi = 12\pi$.

2 Practice

2.1 Convert 120° to radians (exact). [1]

2.2 A sector has $r = 5$ cm and $\theta = 0.8$ rad. Find its arc length. [1]

3 Exam-style questions

3.1 The area of a sector with $r = 4$ and $\theta = \frac{\pi}{6}$ is: [1]

- **A** $\dfrac{2\pi}{3}$
- **B** $\dfrac{4\pi}{3}$
- **C** $\dfrac{\pi}{3}$
- **D** 2π

3.2 A sector OAB has centre O, radius 10 cm and angle $AOB = 1.2$ radians.

(a) Find the perimeter of the sector. [3]

(b) Find the area of the **segment** between the chord AB and the arc. [3]

3.3 A sector has perimeter 20 cm and area 16 cm^2. Show that the radius r satisfies $r^2 - 10r + 16 = 0$, and hence find the two possible radii. [4]

Solutions

2.1 $120 \times \dfrac{\pi}{180} = \dfrac{2\pi}{3}$.

2.2 $s = r\theta = 5 \times 0.8 = 4$ cm.

3.1 **B**. $A = \frac{1}{2}(16)\dfrac{\pi}{6} = \dfrac{4\pi}{3}$.

3.2 (a) arc $= 10 \times 1.2 = 12$; perimeter $= 12 + 10 + 10 = 32$ cm.
(b) segment $= \frac{1}{2}(10^2)(1.2 - \sin 1.2) = 50(1.2 - 0.9320) = 50(0.2680) = 13.4$ cm^2.

3.3 perimeter: $2r + r\theta = 20$; area: $\frac{1}{2}r^2\theta = 16 \Rightarrow r\theta = \dfrac{32}{r}$. Substitute: $2r + \dfrac{32}{r} = 20 \Rightarrow 2r^2 - 20r + 32 = 0 \Rightarrow r^2 - 10r + 16 = 0$. Factorise $(r-2)(r-8) = 0 \Rightarrow r = 2$ or $r = 8$.

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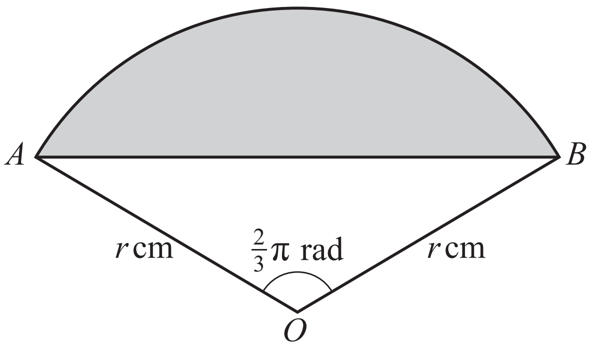

The diagram shows a sector of a circle with centre O and radius r cm. The shaded region is bounded by the chord AB and the arc AB. The size of angle AOB is $\frac{2}{3}\pi$ radians.

(a) Show that the area of the shaded region is approximately $0.614r^2$ cm^2. [2]

It is given that the radius of the circle is increasing at a rate of $0.4\,\text{cm}\,\text{s}^{-1}$.

(b) **(i)** Find the rate of increase of the area of the shaded region at the instant when $r = 20$. Give your answer correct to 2 significant figures. [3]

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(ii) Find the rate of increase of the length of the arc AB. Give your answer correct to 2 significant figures. [3]

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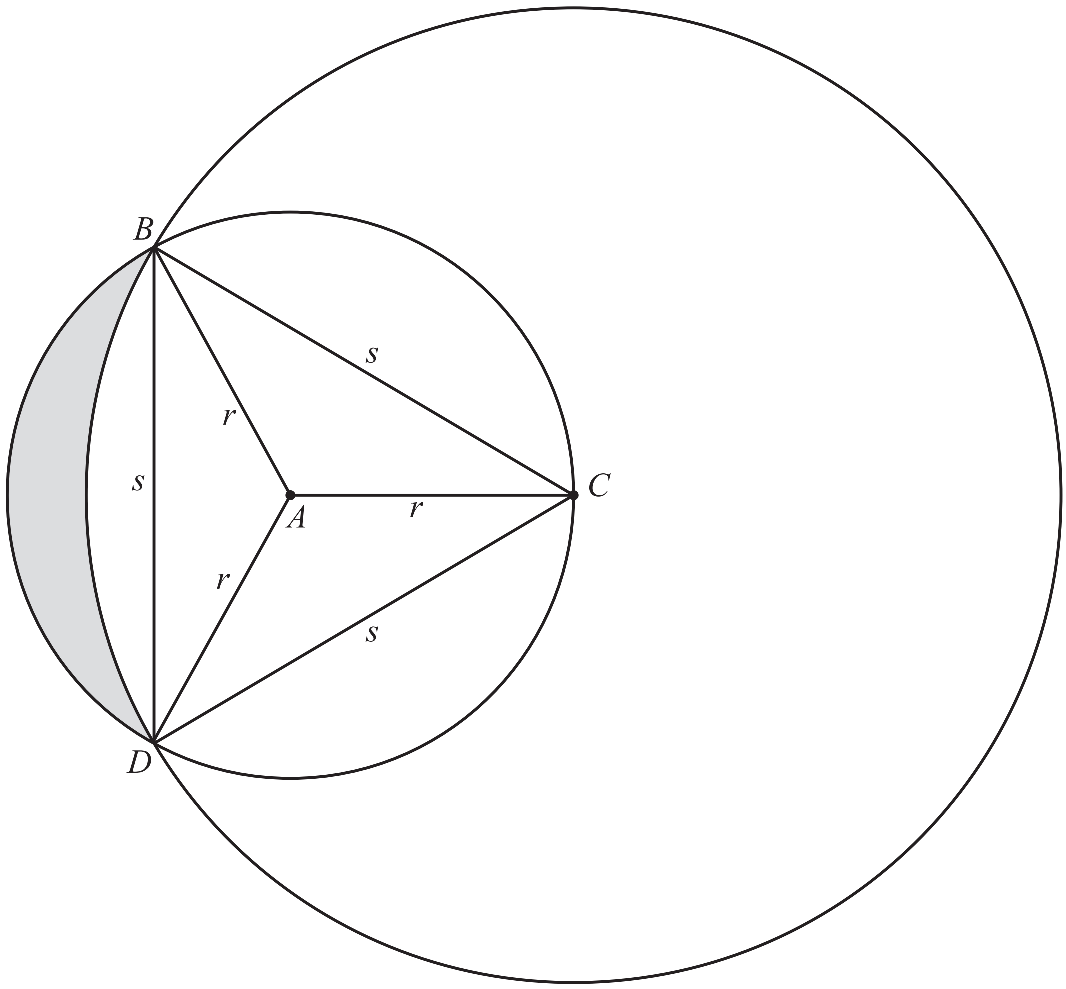

The diagram shows a circle with centre A and radius r passing through points B, C and D. A larger circle of radius s has centre C and passes through B and D. The length BD is also s.

(a) Show that $s = \sqrt{3}r$. [2]

(b) Find an expression for the area of the shaded region. Give your answer in the form $(a+b\pi)r^2$, where a and b are constants to be found. [7]

Additional page

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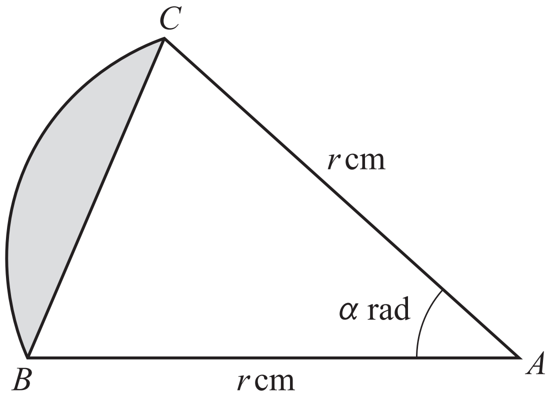


The diagram shows a sector ABC of a circle with centre A and radius r cm. The angle BAC is α radians, where $0 < \alpha < \frac{1}{2}\pi$.

(a) It is given that the area of the triangle ABC is $4\,\text{cm}^2$ and the area of the sector ABC is $8\alpha\,\text{cm}^2$.

Find the exact area of the shaded segment. [4]

(b) It is given instead that the length of the chord BC is $\frac{1}{\sqrt{2}}r$ cm but the area of the triangle ABC is still $4\,\text{cm}^2$.

Find the area of the shaded segment. Give your answer correct to 3 significant figures. [4]

A-LEVEL MATHEMATICS • EXERCISE SHEET

1.5 Trigonometry

Name: ______________ Class: __________ Date: ____________ **Total: 12 marks**

KEY WORDS trigonometric equation 三角方程 • sine 正弦 • function 函数 • identity 恒等式 • trigonometry 三角学

Objective

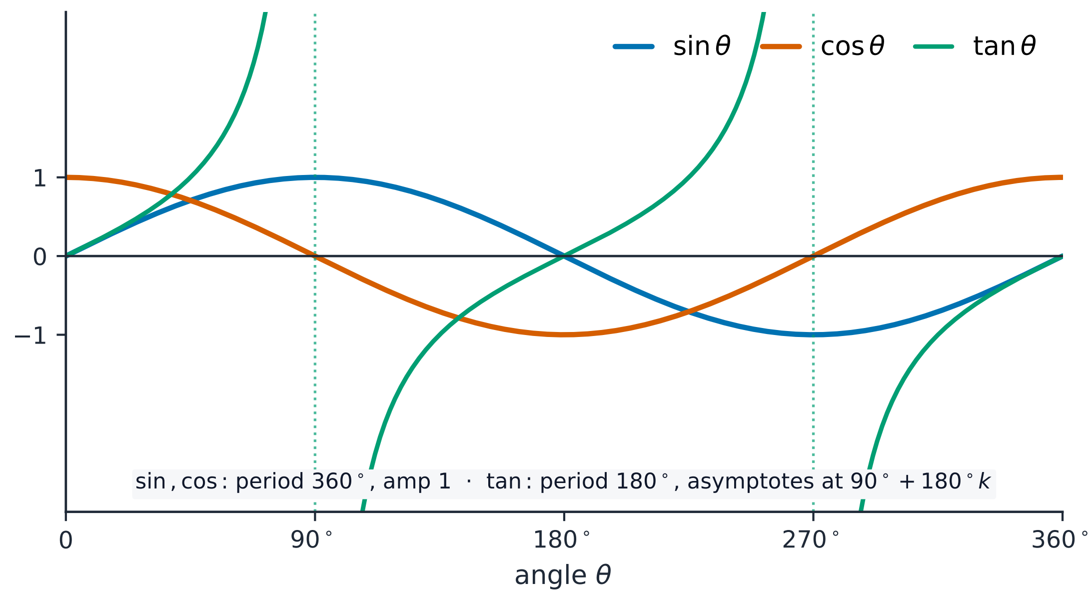
Build the skills to answer exam questions on **trigonometry** 三角学 — exact values, the **identities** 恒等式, and solving **trigonometric equations** 三角方程 in a given interval.

You must be able to:

- recall exact values of sin, cos, tan for $30°, 45°, 60°$
- use $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$
- solve equations (including quadratics in a trig function) over an interval

1 Worked examples

■ Solving a trig equation



$\sin$ *and* $\cos$ *stay between* -1 *and* 1*;* $\tan$ *shoots off at* $90°$ *and* $270°$

Solve $6\sin\theta = 1 + \dfrac{2}{\sin\theta}$ **for** $-180° < \theta < 180°$**.**

$\times\sin\theta$: $6\sin^2\theta - \sin\theta - 2 = 0 \Rightarrow (3\sin\theta - 2)(2\sin\theta + 1) = 0.$ $\sin\theta = \frac{2}{3} \Rightarrow \theta =$

$41.8°, 138.2°$; $\sin\theta = -\frac{1}{2} \Rightarrow \theta = -30°, -150°$.

2 Practice

2.1 Write down the exact value of $\cos 30°$. [1]

2.2 Use an identity to write $\dfrac{\sin\theta}{\cos\theta}$ as a single function. [1]

3 Exam-style questions

3.1 The number of solutions of $\sin\theta = 0.4$ for $0° \leqslant \theta \leqslant 360°$ is: [1]

- **A** 0
- **B** 1
- **C** 2
- **D** 4

3.2 Solve $3\cos^2\theta + 2\sin\theta - 3 = 0$ for $0° \leqslant \theta \leqslant 360°$. [5]

3.3 Solve $2\sin 2x = 1$ for $0° \leqslant x \leqslant 180°$. [4]

Solutions

2.1 $\dfrac{\sqrt{3}}{2}$.

2.2 $\tan\theta$.

3.1 **C**. The sine curve gives two solutions in one full turn for $0 < 0.4 < 1$.

3.2 use $\cos^2\theta = 1 - \sin^2\theta$: $3(1 - \sin^2\theta) + 2\sin\theta - 3 = 0 \Rightarrow -3\sin^2\theta + 2\sin\theta = 0$; $\sin\theta(3\sin\theta - 2)\cdot(-1) = 0 \Rightarrow \sin\theta = 0$ or $\sin\theta = \frac{2}{3}$; $\sin\theta = 0 \Rightarrow \theta = 0°, 180°, 360°$; $\sin\theta = \frac{2}{3} \Rightarrow \theta = 41.8°, 138.2°$.

3.3 $\sin 2x = \frac{1}{2}$; for $0° \leqslant 2x \leqslant 360°$, $2x = 30°, 150°$; $x = 15°, 75°$.

5 **(a)** Show that $\tan^4\theta - 1 \equiv \dfrac{1-2\cos^2\theta}{\cos^4\theta}$. [3]

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(b) Hence solve the equation $\cos^2\theta(\tan^4\theta - 1) = 7$ for $0° < \theta < 180°$. [4]

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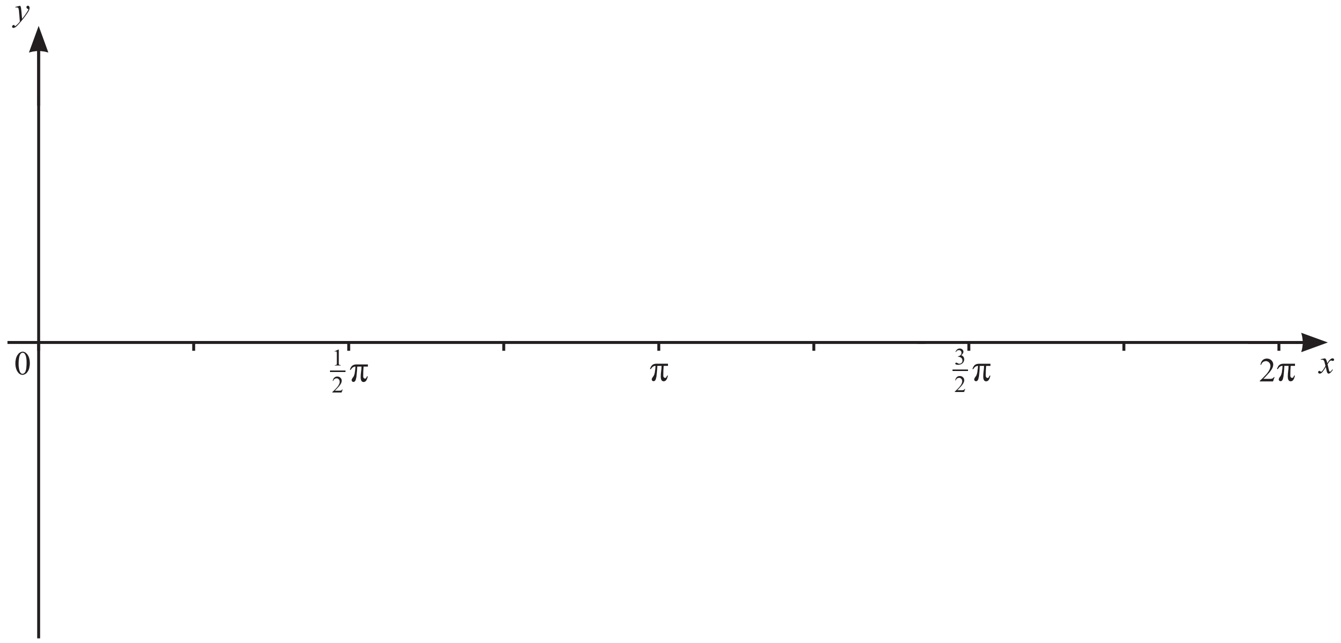
6 **(a)** Sketch the graph of $y = 3\sin x + 2$ for $0 \leqslant x \leqslant 2\pi$. [2]



(b) Determine the number of solutions in the interval $0 \leqslant x \leqslant 2\pi$ of each of the following equations.

(i) $3\sin x + 2 = x$ [1]

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(ii) $3\sin x + 2 = 5 - x$ [1]

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(c) Solve the equation $3\sin x + 2 = 5\cos^2 x - 1$ for $0 \leqslant x \leqslant 2\pi$. [5]

1 Solve the equation $6\sin\theta = 1 + \frac{2}{\sin\theta}$ for $-180° < \theta < 180°$. [4]

5 The equation of a curve is $y = 4\cos 2x + 3$ for $0 \leqslant x \leqslant 2\pi$.

(a) State the greatest and least possible values of y. [2]

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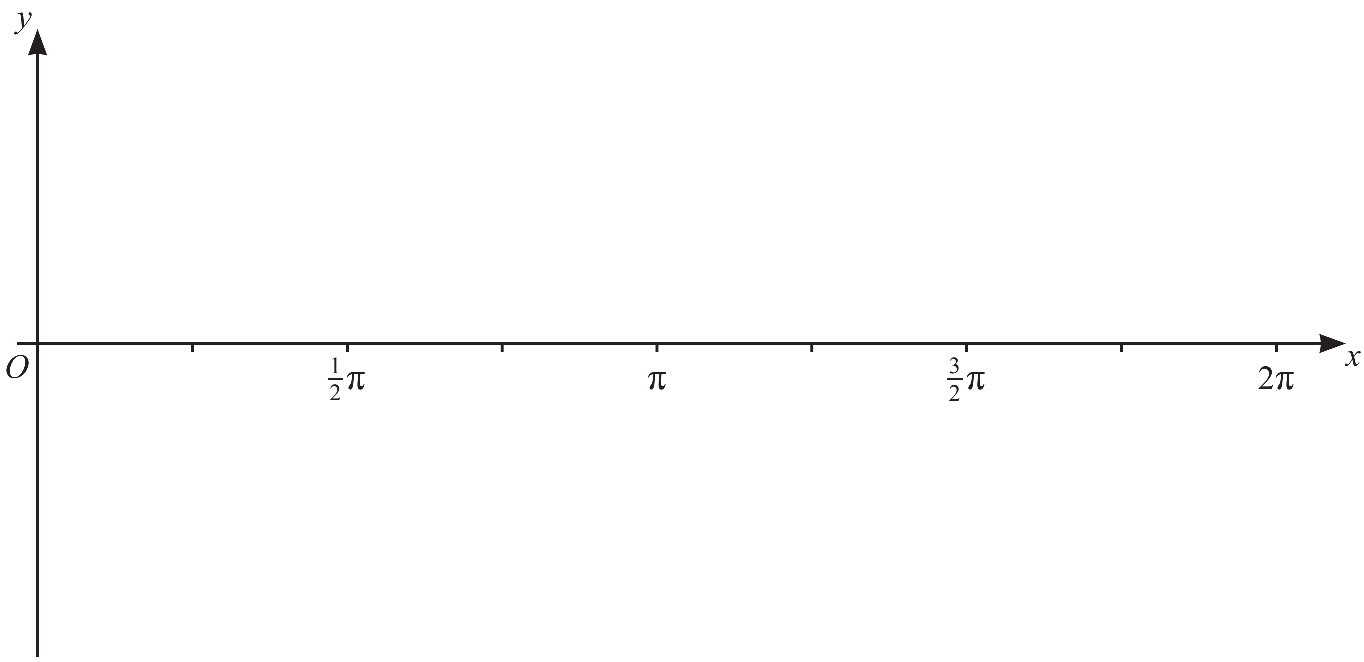
(b) Sketch the curve. [2]



(c) Hence determine the number of solutions of the equation $4\cos 2x + 3 = 2x - 1$ for $0 \leqslant x \leqslant 2\pi$. [1]

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7 **(a)** Prove the identity $\dfrac{\tan\theta + 7}{\tan^2\theta - 3} \equiv \dfrac{\sin\theta\cos\theta + 7\cos^2\theta}{1 - 4\cos^2\theta}$. [3]

(b) Hence solve the equation $\dfrac{\sin\theta\cos\theta+7\cos^2\theta}{1-4\cos^2\theta}=\dfrac{5}{\tan\theta}$ for $0° \leqslant \theta \leqslant 180°$. [4]

1.6 Series

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

KEY WORDS progression 数列 • geometric progression 等比数列 • arithmetic 等差 • sum to infinity 无穷和 • common ratio 公比

Objective

Build the skills to answer exam questions on **series** 级数 — the **binomial expansion** 二项展开式, **arithmetic** 等差 and **geometric progressions** 等比数列, and the **sum to infinity** 无穷和.

You must be able to:

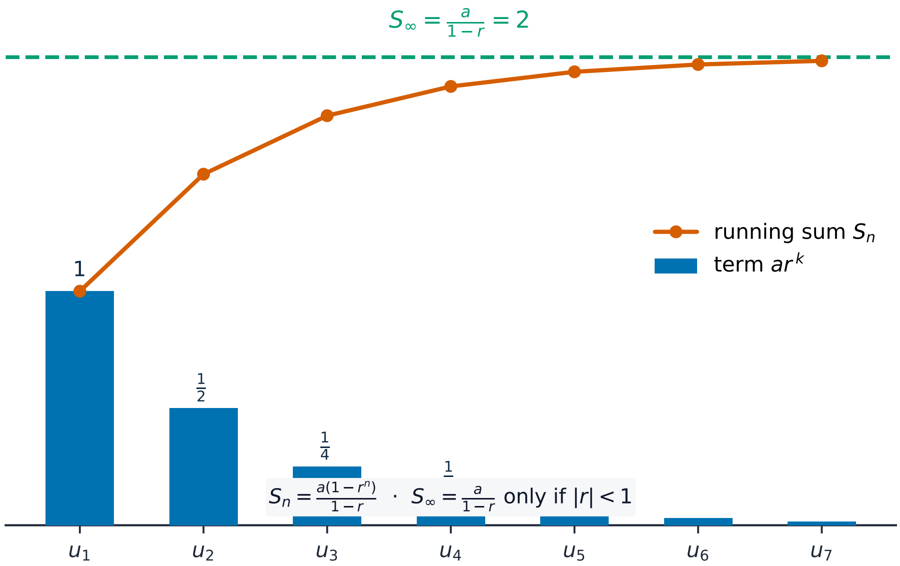
- expand $(a+b)^n$ and find a specific term/coefficient
- use the AP and GP formulas for the nth term and the sum
- use $S_\infty = \dfrac{a}{1-r}$ when $|r| < 1$

1 Worked examples

■ Binomial & GP

First three terms of $(2-px)^5$**:**

$$2^5 + \binom{5}{1}2^4(-px) + \binom{5}{2}2^3(-px)^2 = 32 - 80px + 80p^2x^2.$$



A convergent GP ($|r| < 1$): the terms shrink and the running sum approaches
$$S_\infty = \frac{a}{1-r}$$

A GP has third term 18 and $S_3 = 26$, with $r < 0$. From $ar^2 = 18$ and $a(1 + r + r^2) = 26$: $8r^2 - 18r - 18 = 0$, so $r = -\frac{3}{4}$, $a = 32$, and $S_\infty = \dfrac{32}{1 + \frac{3}{4}} = \dfrac{128}{7}$.

2 Practice

2.1 Find the 10th term of the AP $3, 7, 11, \ldots$. [2]

2.2 State the condition on r for a GP to have a sum to infinity. [1]

3 Exam-style questions

3.1 The coefficient of x^2 in $(1 + 2x)^6$ is: [1]

- **A** 12
- **B** 30
- **C** 60
- **D** 15

3.2 The first term of a GP is 24 and the sum to infinity is 40.

(a) Find the common ratio. [3]

(b) Find the sum of the first three terms. [2]

3.3 An AP has first term 5 and common difference 3. The sum of the first n terms is 440. Find n. [4]

Solutions

2.1 $u_{10} = a + 9d = 3 + 9(4) = 39$.

2.2 $|r| < 1$.

3.1 C. $\binom{6}{2}(2)^2 = 15 \times 4 = 60$.

3.2 (a) $S_\infty = \dfrac{a}{1-r} = 40 \Rightarrow \dfrac{24}{1-r} = 40 \Rightarrow 1 - r = 0.6$; $r = 0.4$.
(b) $S_3 = \dfrac{24(1-0.4^3)}{1-0.4} = \dfrac{24(0.936)}{0.6} = 37.44$.

3.3 $S_n = \dfrac{n}{2}(2a + (n-1)d) = 440 \Rightarrow \dfrac{n}{2}(10 + 3(n-1)) = 440$; $3n^2 + 7n - 880 = 0$; $(3n+55)(n-16) = 0$; $n = 16$ (reject negative).

2 A geometric progression has first term a and common ratio $\cos\theta$, where $0 < \theta < \frac{1}{2}\pi$. It is given that the second term is 8 and the fifth term is $\frac{1}{8}$.

(a) Find the value of θ. Give your answer correct to 3 significant figures. [3]

(b) Find the exact value of the sum to infinity. [2]

3 In the expansion of

$$(px+3)^5 - \left(x^3 + \frac{p}{x}\right)^4,$$

the coefficient of x^4 is 216.

Find the value of the positive constant p. [5]

9 An arithmetic progression has first term 2 and common difference d. The sum of the first n terms is denoted by S_n.

(a) It is given that $(S_2 - 1)$, S_4, S_9 are the first three terms of a second arithmetic progression.

Find the value of d. [4]

(b) Hence find the difference between the values of the 15th terms of the two arithmetic progressions. [4]

2 Find the term independent of x in the expansion of $\left(2x^2 - \frac{3}{x}\right)$. [3]

8 The first three terms of a geometric progression are a, b and c respectively, where a, b and c are positive constants. The first three terms of an arithmetic progression are a, b and $-3c$ respectively.

(a) Show that $a^2 - 10ac + 9c^2 = 0$. [3]

It is now given that $a = 9$ and c takes the smaller of its two possible values.

(b) **(i)** Find the sum to infinity of the geometric progression. [5]

(ii) Find the sum of the first 20 terms of the arithmetic progression. [3]

3 The third term of a geometric progression is 18 and the sum of the first three terms is 26. It is given that the common ratio is negative.

(a) Find the tenth term of the progression. Give your answer correct to 3 significant figures. [5]

(b) Find the exact value of the sum to infinity of the progression. [2]

5 **(a)** Find the first three terms, in ascending powers of x, in the expansion of each of the following expressions.

(i) $(2-px)^5$ [2]

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(ii) $\left(1-\frac{1}{2}x\right)^4$ [2]

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(b) Given that the coefficient of x^2 in the expansion of $(2-px)^5\left(1-\frac{1}{2}x\right)^4$ is 93, find the possible values of the constant p. [3]

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3 The coefficient of x^7 in the expansion of $\left(px^2 + \frac{4}{p}x\right)$ is 1280.

Find the value of the constant p. [4]

10 **(a)** The first, second and third terms of an arithmetic progression are $4k$, k^2 and $8k$ respectively, where k is a non-zero constant.

(i) Find the value of k. [2]

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(ii) Find the sum of the first 20 terms of the progression. [3]

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(b) The fourth and sixth terms of a geometric progression are 36 and 6 respectively. The common ratio of the progression is positive.

Find the sum to infinity of the progression. Give your answer in the form $\frac{a}{\sqrt{b}-c}$, where a, b and c are integers. [5]

1.7 Differentiation

Name: ______ Class: ______ Date: ______ **Total: 15 marks**

KEY WORDS stationary point 驻点 • normal 法线 • chain rule 链式法则 • tangents and normals 切线与法线 • gradient 斜率

Objective

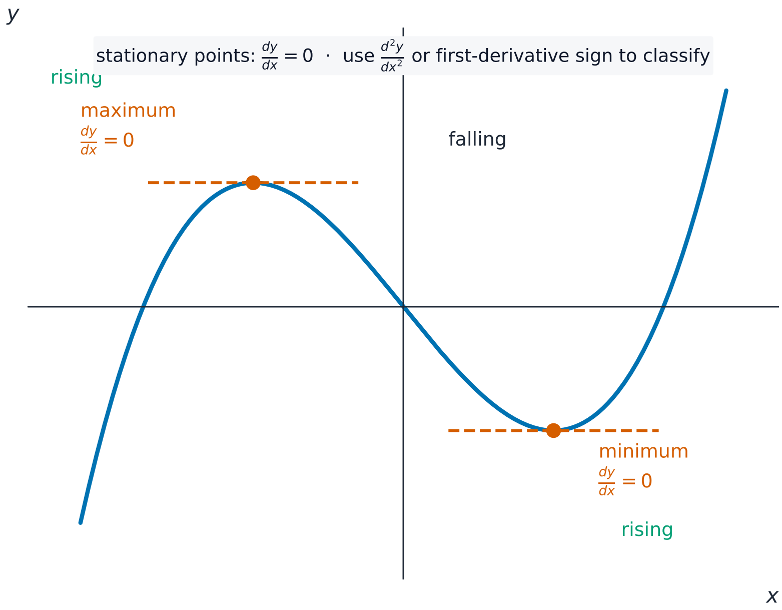
Build the skills to answer exam questions on **differentiation 微分** — the power and **chain rule 链式法则**, **tangents and normals 切线与法线**, increasing/decreasing, and **stationary points 驻点**.

You must be able to:

- differentiate powers and use the chain rule
- find the equations of a tangent and a normal
- find and classify stationary points (max/min) with $f''(x)$

1 Worked examples

■ Stationary points



$$\textit{At a stationary point } \frac{dy}{dx} = 0;\ f''(x) > 0 \to \textit{minimum},\ f''(x) < 0 \to \textit{maximum}$$

$y = 4x^{1/2} - x$ **has a maximum at** $x = a$**. Find** a**.**

$$\frac{dy}{dx} = 2x^{-1/2} - 1 = 0 \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4,\ \text{so } a = 4.$$

Chain rule: $\frac{d}{dx}(2x-1)^4 = 4(2x-1)^3 \cdot 2 = 8(2x-1)^3$.

2 Practice

2.1 Differentiate $y = 3x^4 - \frac{2}{x}$. [2]

2.2 Find $\frac{dy}{dx}$ when $y = (3x+1)^5$. [2]

3 Exam-style questions

3.1 The gradient of $y = x^3 - 2x$ at $x = 2$ is: [1]

- **A** 4
- **B** 6
- **C** 10
- **D** 12

3.2 A curve has equation $y = x^3 - 6x^2 + 9x + 1$.

(a) Find the coordinates of the two stationary points. [4]

(b) Determine the nature of each stationary point. [2]

3.3 The curve $y = \dfrac{8}{x}$ passes through $P(2, 4)$. Find the equation of the **normal** to the curve at P. [4]

Solutions

2.1 $\dfrac{dy}{dx} = 12x^3 + \dfrac{2}{x^2}$.

2.2 $\dfrac{dy}{dx} = 5(3x+1)^4 \cdot 3 = 15(3x+1)^4$.

3.1 **C**. $\dfrac{dy}{dx} = 3x^2 - 2$; at $x = 2$, $= 12 - 2 = 10$.

3.2 (a) $\dfrac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$; $x = 1, 3$; $y(1) = 5$, $y(3) = 1$, so $(1,5)$ and $(3,1)$.
(b) $\dfrac{d^2y}{dx^2} = 6x - 12$; at $x = 1$, $= -6 < 0 \rightarrow$ maximum; at $x = 3$, $= 6 > 0 \rightarrow$ minimum.

3.3 $\dfrac{dy}{dx} = -\dfrac{8}{x^2}$; at $x = 2$, gradient $= -2$; normal gradient $= \frac{1}{2}$; $y - 4 = \frac{1}{2}(x-2) \Rightarrow y = \frac{1}{2}x + 3$.

11 A curve passes through the point $P(4, 3)$ and is such that

$$\frac{dy}{dx} = \frac{8}{x^2} - \frac{10}{(2x-3)^2}.$$

(a) Find the equation of the normal to the curve at P. Give your answer in the form $y = mx + c$. [3]

(b) Find the rate of change of the gradient of the curve when $x = 4$. [3]

(c) Given that the curve also passes through the point $(-1, q)$, find the value of q. [5]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

4 The equation of a curve is such that $\frac{dy}{dx} = kx^3 + \frac{2}{x^2}$, where k is a constant. The curve passes through the point S (2, 20) and the gradient of the curve at S is $\frac{65}{2}$.

(a) Find the value of k. [1]

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(b) The coordinates of a point T on the curve are (1, t).

Find the value of t. [5]

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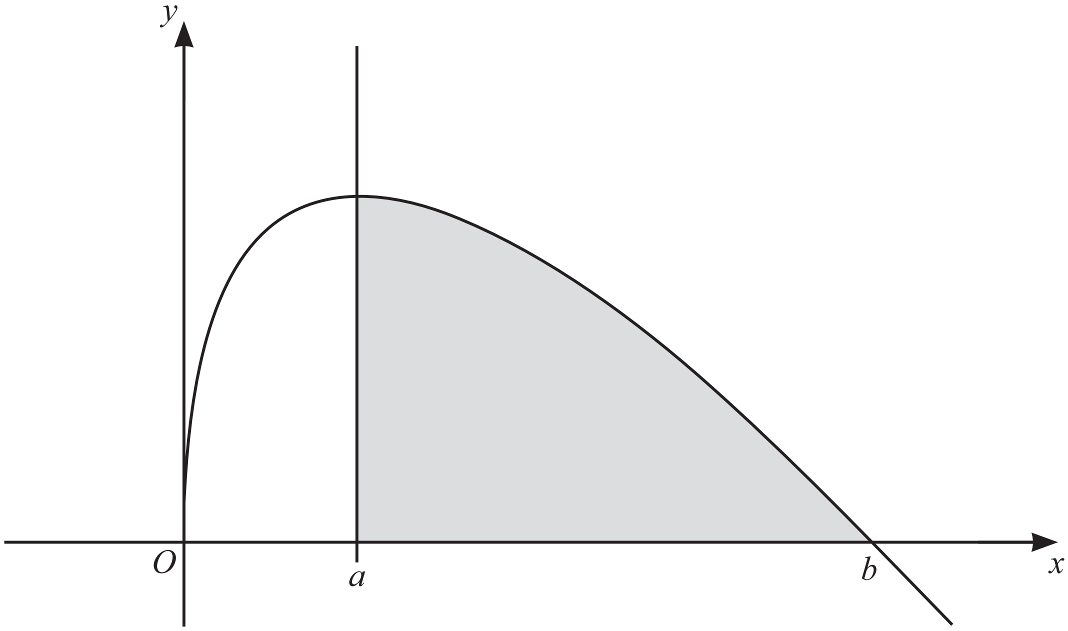

The equation of a curve is $y = 4x^{\frac{1}{2}} - x$. The curve has a maximum point when $x = a$ and crosses the x-axis at the point with coordinates $(b, 0)$, where $b > 0$. The shaded region is bounded by the curve, the line $x = a$ and the x-axis (see diagram).

(a) Find the value of a. [3]

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(b) Find the exact area of the shaded region. [5]

9 The function f is defined by $\mathrm{f}(x) = \dfrac{4}{(3x-6)^2} + \dfrac{1}{(3x-6)^3}$ for $x > 2$.

(a) Find an expression for $\mathrm{f}'(x)$ and hence determine whether f is an increasing function, a decreasing function or neither. [4]

(b) State whether f^{-1} exists. Give a reason for your answer. [1]

The function g is defined by $g(x) = 4x - 3$ for $x > a$.

(c) Find the range of g in terms of the constant a. [1]

(d) Find the set of values of a for which the composite function fg exists. [2]

2 The equation of a curve is such that $\frac{dy}{dx} = 4(2x-5)^3 - 9x^{\frac{1}{2}}$. The curve passes through the point $A\left(4, -\frac{11}{2}\right)$.

(a) Find the gradient of the normal to the curve at the point A. [2]

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(b) Find the equation of the curve. [4]

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7 The equation of a curve is $y = 4x^2 + \frac{9}{x^2} - 8$.

(a) A point P is moving along the curve in such a way that its y-coordinate is decreasing at 5 units per second.

Find the rate at which the x-coordinate of point P is changing when $x = 2$. [4]

(b) Find the coordinates of the stationary points of the curve and determine their nature. [5]

4 A point P is moving along the curve with equation $y = ax^{\frac{3}{2}} - 12x$ in such a way that the x-coordinate of P is increasing at a constant rate of 5 units per second.

(a) Find the rate at which the y-coordinate of P is changing when $x = 9$. Give your answer in terms of the constant a. [3]

(b) Given that the curve has a minimum point when $x = \frac{1}{4}$, find the value of a. [2]

9 The equation of a curve is such that $\frac{d^2y}{dx^2} = -\frac{24}{x^3}$. It is given that the curve has a stationary point at $(-2, 19)$.

(a) Find an expression for $\frac{dy}{dx}$. [3]

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(b) Find the x-coordinate of the other stationary point of the curve, and determine the nature of this stationary point. [2]

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(c) Find the equation of the curve. [3]

(d) Find the equation of the normal to the curve at the point where $\frac{dy}{dx} = -\frac{9}{4}$ and x is positive. Express your answer in the form $px + qy + r = 0$, where p, q and r are integers. [4]

1.8 Integration

Name: ____________ Class: ________ Date: __________ **Total: 15 marks**

KEY WORDS definite integral 定积分 • volume of revolution 旋转体体积 • integration 积分 • region 区域

Objective

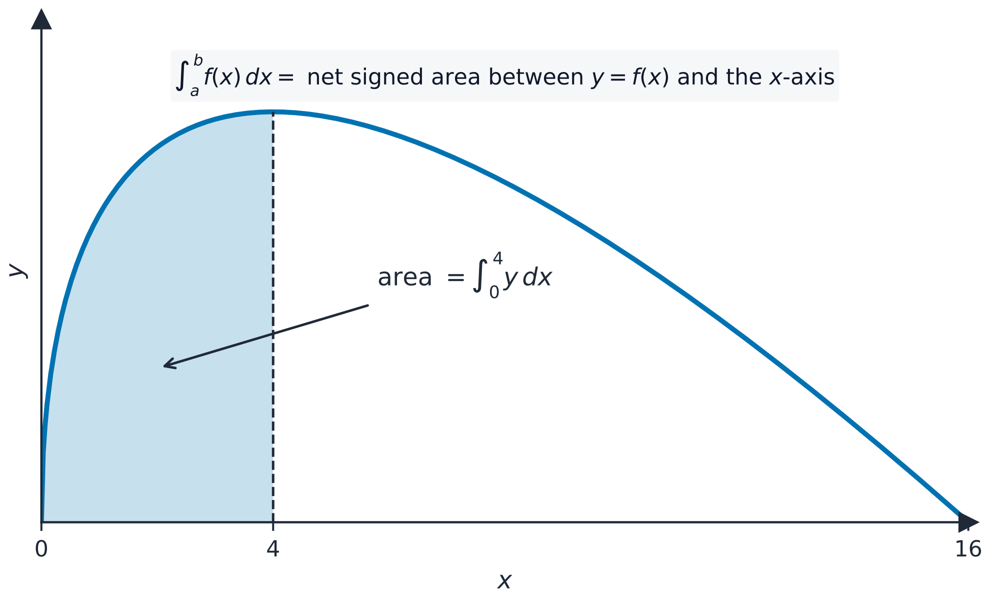
Build the skills to answer exam questions on **integration 积分** — indefinite and **definite integrals 定积分**, area under a curve, and **volume of revolution 旋转体体积**.

You must be able to:

- integrate powers and $(ax+b)^n$, finding $+C$ from a point
- evaluate a definite integral and find an area
- find a volume of revolution with $V = \pi \int y^2\,dx$

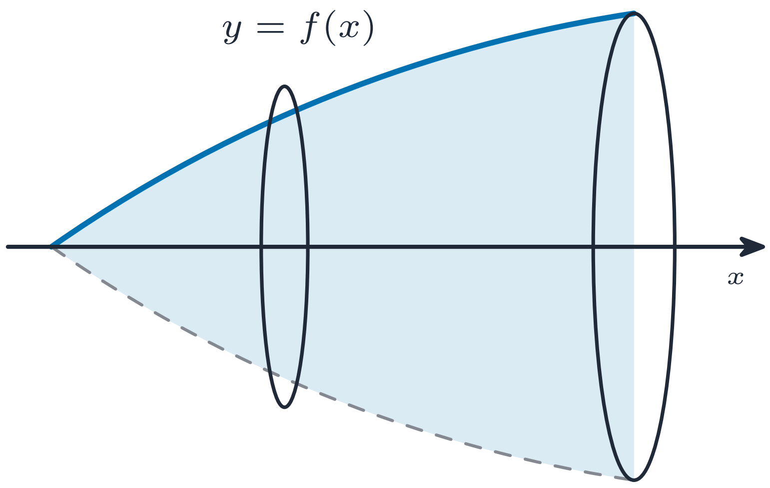
1 Worked examples

Area and volume



The definite integral $\int_p^q y\,dx$ is the shaded area under the curve

$$\int_0^4 (4x^{1/2} - x)\,dx = \left[\tfrac{8}{3}x^{3/2} - \tfrac{x^2}{2}\right]_0^4 = \tfrac{8}{3}(8) - 8 = \tfrac{40}{3}.$$



rotate the region around the x-axis to sweep a solid

Rotating the region about the x-axis: each slice is a disc of area πy^2, so $V = \pi \int y^2\, dx$

2 Practice

2.1 Find $\displaystyle\int (6x^2 - 4x + 1)\, dx$. [2]

2.2 Evaluate $\displaystyle\int_1^2 \frac{1}{x^2}\, dx$. [2]

3 Exam-style questions

3.1 $\int_0^3 2x\,dx$ equals: [1]

- **A** 6
- **B** 9
- **C** 12
- **D** 18

3.2 A curve has $\frac{dy}{dx} = 3x^2 - 4x$ and passes through $(2, 5)$. Find the equation of the curve. [4]

3.3 The region R is bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$.

(a) Find the area of R. [3]

(b) Find the volume when R is rotated through 360° about the x-axis. [3]

Solutions

2.1 $2x^3 - 2x^2 + x + C$.

2.2 $\left[-\dfrac{1}{x}\right]_1^2 = -\dfrac{1}{2} - (-1) = \dfrac{1}{2}$.

3.1 **B**. $[x^2]_0^3 = 9$.

3.2 $y = \displaystyle\int (3x^2 - 4x)\,dx = x^3 - 2x^2 + C$; use $(2, 5)$: $5 = 8 - 8 + C \Rightarrow C = 5$; $y = x^3 - 2x^2 + 5$.

3.3 (a) $\displaystyle\int_0^4 x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^4 = \tfrac{2}{3}(8) = \dfrac{16}{3}$.

(b) $V = \pi \displaystyle\int_0^4 (\sqrt{x})^2\,dx = \pi \int_0^4 x\,dx = \pi \left[\tfrac{x^2}{2}\right]_0^4 = 8\pi$.

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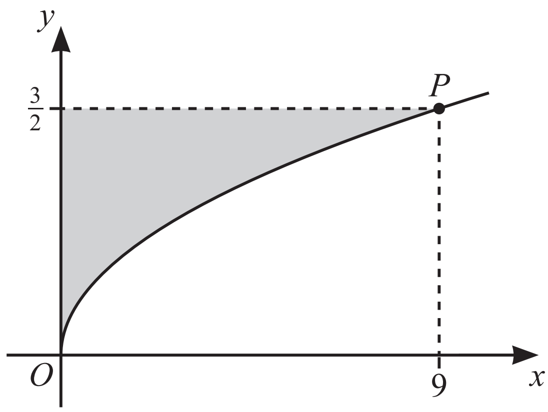


The diagram shows the curve with equation $y = \frac{1}{2}\sqrt{x}$ and the point P with coordinates $\left(9, \frac{3}{2}\right)$. The shaded region is bounded by the curve and the lines $x = 0$ and $y = \frac{3}{2}$.

(a) Find the area of the shaded region. [3]

(b) The shaded region is rotated through 360° about the ***y*-axis**.

Find the exact volume of the solid produced. [4]

11 A curve passes through the point $P(4, 3)$ and is such that

$$\frac{dy}{dx} = \frac{8}{x^2} - \frac{10}{(2x-3)^2}.$$

(a) Find the equation of the normal to the curve at P. Give your answer in the form $y = mx + c$. [3]

(b) Find the rate of change of the gradient of the curve when $x = 4$. [3]

(c) Given that the curve also passes through the point $(-1, q)$, find the value of q. [5]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

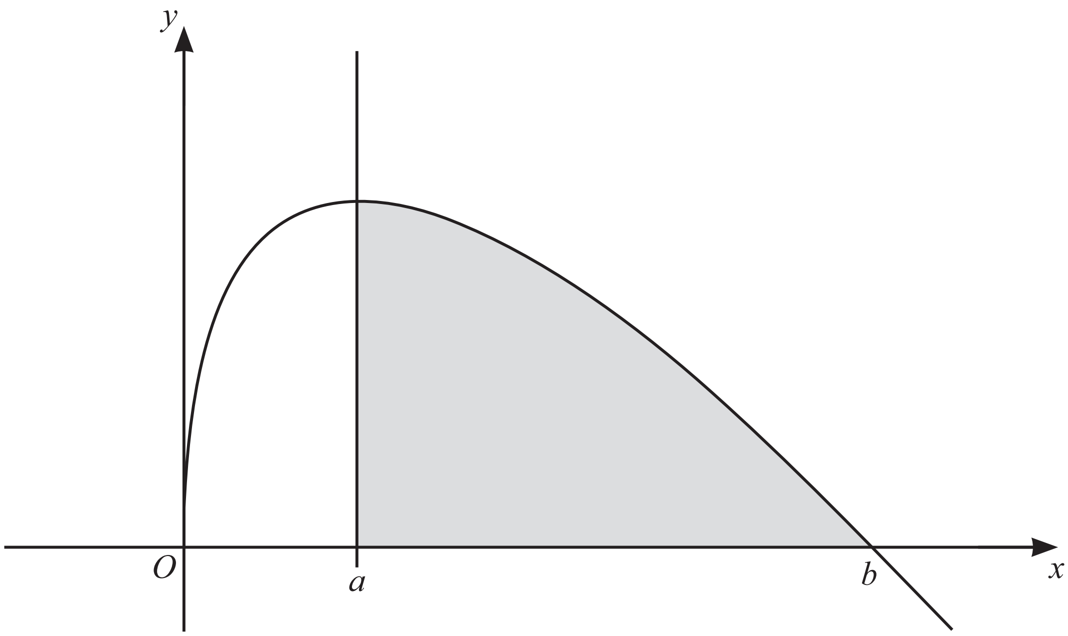
4 The equation of a curve is such that $\frac{dy}{dx} = kx^3 + \frac{2}{x^2}$, where k is a constant. The curve passes through the point $S\,(2, 20)$ and the gradient of the curve at S is $\frac{65}{2}$.

(a) Find the value of k. [1]

(b) The coordinates of a point T on the curve are $(1, t)$.

Find the value of t. [5]

5



The equation of a curve is $y = 4x^{\frac{1}{2}} - x$. The curve has a maximum point when $x = a$ and crosses the x-axis at the point with coordinates $(b, 0)$, where $b > 0$. The shaded region is bounded by the curve, the line $x = a$ and the x-axis (see diagram).

(a) Find the value of a. [3]

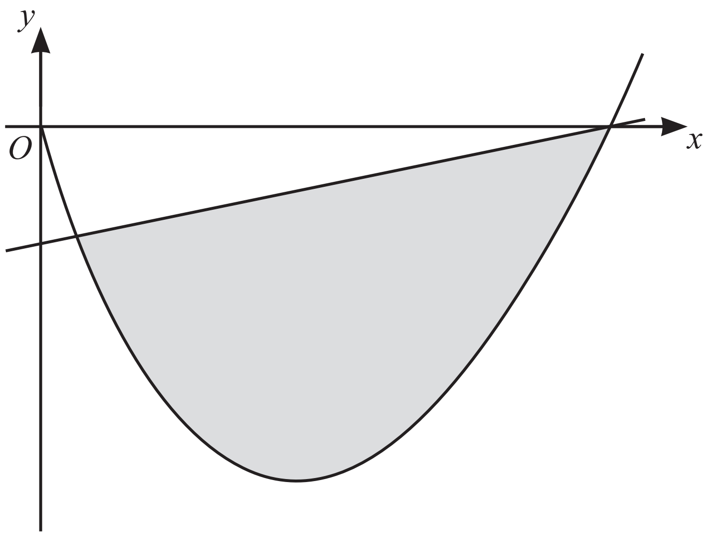
(b) Find the exact area of the shaded region. [5]

2 The equation of a curve is such that $\frac{dy}{dx} = 4(2x-5)^3 - 9x^{\frac{1}{2}}$. The curve passes through the point $A\left(4, -\frac{11}{2}\right)$.

(a) Find the gradient of the normal to the curve at the point A. [2]

(b) Find the equation of the curve. [4]

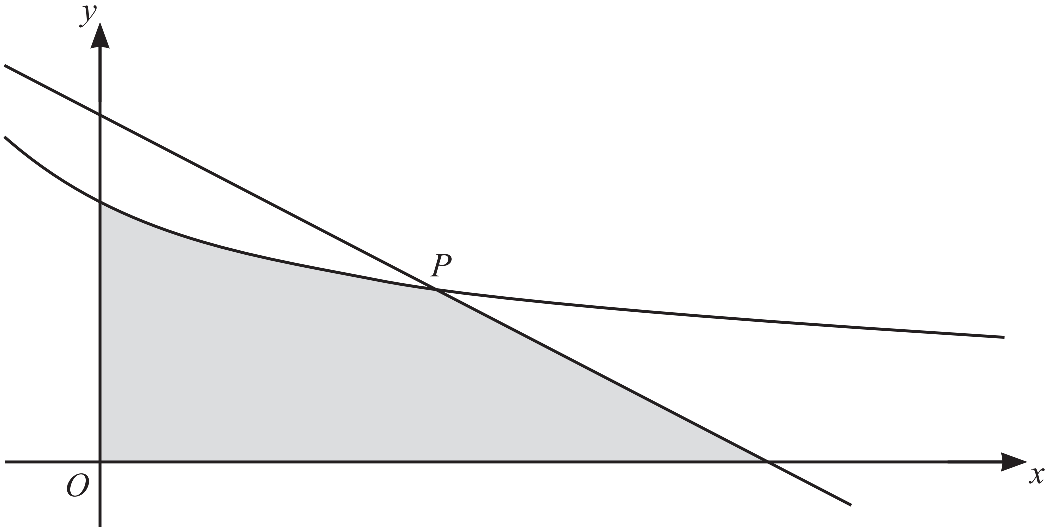
4



The diagram shows the curve with equation $y = 5x^{\frac{3}{2}} - 20x$ and the line with equation $y = x - 16$.
The x-coordinates of the points of intersection of the curve and line are 1 and 16.

Find the area of the shaded region between the curve and the line. [5]

6



The diagram shows the curve with equation $y = \dfrac{9}{(5x+4)^{\frac{1}{2}}}$ and the line $y = 6 - 3x$. The line and the curve intersect at the point P which has y-coordinate 3.

Find the area of the shaded region. [6]

9 The equation of a curve is such that $\frac{d^2y}{dx^2} = -\frac{24}{x^3}$. It is given that the curve has a stationary point at $(-2, 19)$.

(a) Find an expression for $\frac{dy}{dx}$. [3]

(b) Find the x-coordinate of the other stationary point of the curve, and determine the nature of this stationary point. [2]

(c) Find the equation of the curve. [3]

(d) Find the equation of the normal to the curve at the point where $\frac{dy}{dx} = -\frac{9}{4}$ and x is positive. Express your answer in the form $px + qy + r = 0$, where p, q and r are integers. [4]