A-LEVEL MATHEMATICS • EXERCISE SHEET

6.3 Continuous random variables

Name: ____________ Class: ________ Date: __________ **Total: 17 marks**

KEY WORDS probability density function 概率密度函数 • median 中位数
• continuous random variable 连续型随机变量 • cumulative distribution function 累积分布函数
• percentiles 百分位数

Objective

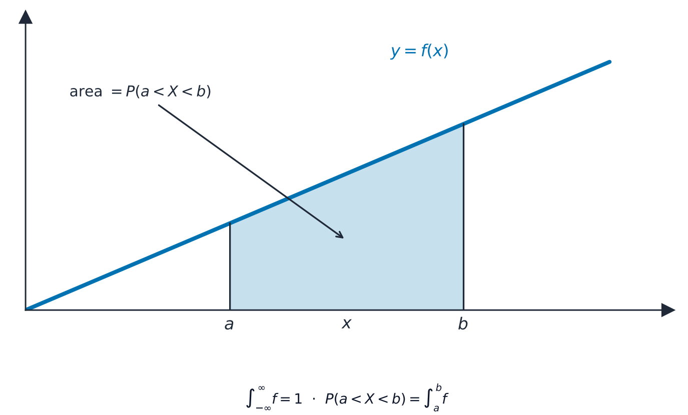
Build the skills to answer exam questions on **continuous random variables** 连续型随机变量 — the **probability density function** 概率密度函数 $f(x)$, finding probabilities and $E(X)$, and the **cumulative distribution function** 累积分布函数 $F(x)$.

You must be able to:

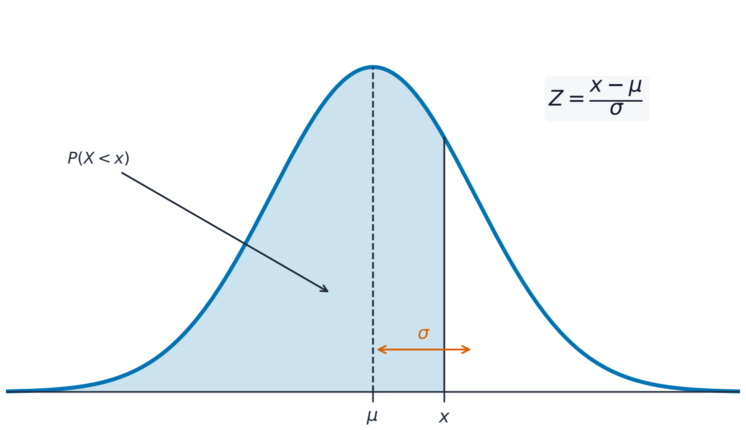
- use $\int f(x)\,dx = 1$ to find a constant; find $P(a < X < b) = \int_a^b f\,dx$
- find $E(X) = \int x f(x)\,dx$ (and $\mathrm{Var}(X)$)
- find $F(x)$ and use it for the median/percentiles

1 Worked examples

■ Density and expectation



$P(a < X < b)$ is the area under $f(x)$ between a and b; the total area is 1



The normal distribution is a continuous random variable

$f(x) = \frac{1}{2}x$ **for** $0 \leqslant x \leqslant 2$: $E(X) = \int_0^2 x \cdot \frac{1}{2}x\,dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$.

Median m**:** solve $F(m) = 0.5$.

2 Practice

2.1 $f(x) = kx$ for $0 \leqslant x \leqslant 2$. Show that $k = \frac{1}{2}$. [2]

2.2 For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $P(X < 1)$. [2]

3 Exam-style questions

3.1 For a valid pdf, $\int_{-\infty}^{\infty} f(x)\,dx$ must equal: [1]

- **A** 0
- **B** 0.5
- **C** 1
- **D** ∞

3.2 A continuous random variable has $f(x) = \frac{3}{4}(1-x^2)$ for $-1 \leqslant x \leqslant 1$ (and 0 elsewhere).

(a) Verify that $\int_{-1}^{1} f(x)\,dx = 1$. [3]

(b) Find $P(X > 0.5)$. [3]

3.3 A variable has $f(x) = \frac{3}{8}x^2$ for $0 \leqslant x \leqslant 2$.

(a) Find $E(X)$. [3]

(b) Find the median m. [3]

Solutions

2.1 $\int_0^2 kx\,dx = \left[\frac{kx^2}{2}\right]_0^2 = 2k = 1 \Rightarrow k = \frac{1}{2}$.

2.2 $\int_0^1 \frac{1}{2}x\,dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4}$.

3.1 **C**. The total area under a pdf is 1.

3.2 (a) $\int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{3}{4}\left(\frac{2}{3} - (-\frac{2}{3})\right) = \frac{3}{4} \cdot \frac{4}{3} = 1$.

(b) $\int_{0.5}^{1} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{0.5}^{1} = \frac{3}{4}\left(\frac{2}{3} - (0.5 - 0.0417)\right) = \frac{3}{4}(0.6667 - 0.4583) = 0.156$.

3.3 (a) $E(X) = \int_0^2 x \cdot \frac{3}{8}x^2\,dx = \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2 = \frac{3}{8} \cdot 4 = \frac{3}{2}$.

(b) $F(m) = \int_0^m \frac{3}{8}x^2\,dx = \frac{m^3}{8} = 0.5 \Rightarrow m^3 = 4 \Rightarrow m = 1.59$.