A-LEVEL MATHEMATICS • EXERCISE SHEET

3.6 Numerical solution of equations

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

Objective

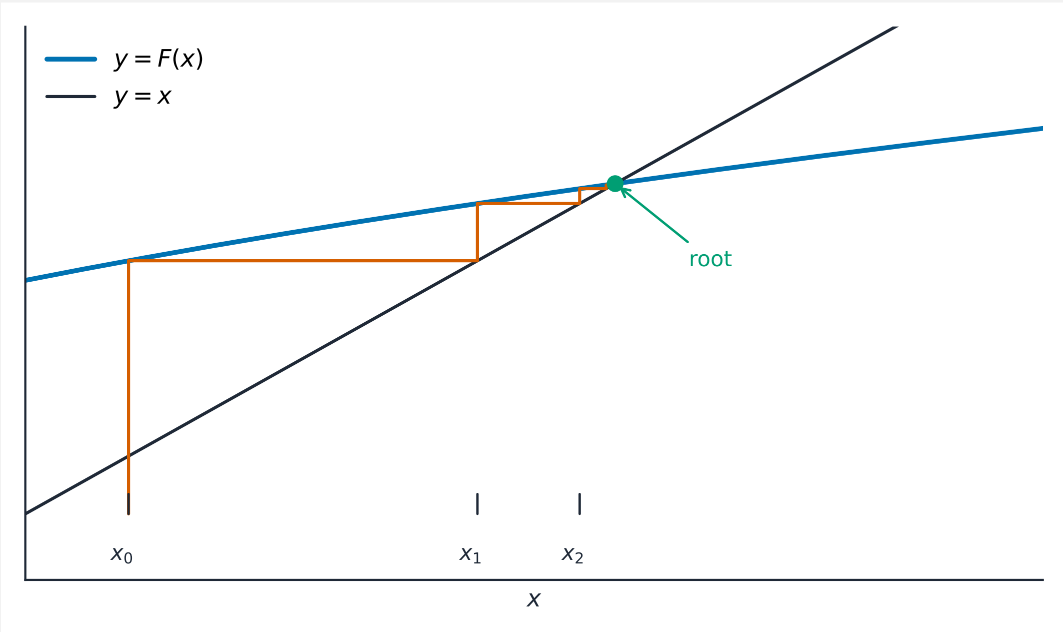
Build the skills to answer exam questions on the **numerical solution of equations** 方程的数值解 in Pure 3 —locating a root by a **sign change** 变号 and finding it by **iteration** 迭代, often after some algebra/calculus.

You must be able to:

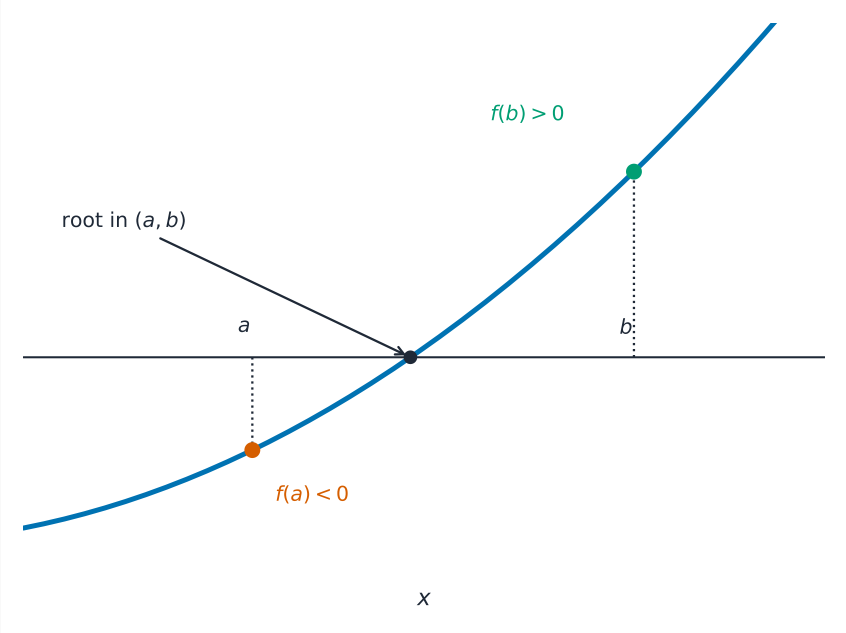
- show a root lies in an interval by a sign change
- show an equation rearranges into a given iterative form
- iterate $x_{n+1} = F(x_n)$ to the required accuracy

1 Worked examples

■ Iteration to a root



Each step: up to $y = F(x)$, across to $y = x$ —the iteration converges to the root



Locate the root first by a sign change

$x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ **from** $x_0 = -1.2$**:** $x_1 = -1.281$, $x_2 = -1.247$, $\cdots \to$ root ≈ -1.26 (3 s.f.).

2 Practice

2.1 $f(x) = e^x - 3x$. Show that a root lies between $x = 1$ and $x = 2$. [2]

2.2 Rearrange $x^3 + x - 5 = 0$ into the form $x = F(x)$ suitable for iteration. [1]

3 Exam-style questions

3.1 An iteration $x_{n+1} = F(x_n)$ from a suitable start **converges** to a root if the values: [1]

- **A** grow without limit
- **B** settle towards a fixed number
- **C** alternate between two values forever
- **D** stay equal to x_0

3.2 The equation $x^3 = 4x + 1$ has a root α near $x = 2$.

(a) Verify there is a root between $x = 2$ and $x = 2.2$. [2]

(b) Show the equation can be written $x = \sqrt[3]{4x + 1}$ and use $x_{n+1} = \sqrt[3]{4x_n + 1}$, $x_0 = 2.1$, to find α to 2 d.p. [4]

3.3 By sketching $y = e^{-x}$ and $y = x$ on the same axes, state the number of roots of $e^{-x} = x$, and give an interval of width 0.5 containing the root. [3]

4 Go further

You are now ready for the real exam questions on this subtopic. Open the **3.6 Numerical solution of equations** past-paper sheet in the Library, or try these in **Practice mode**:

- 9709/31 N25 —Q9 (iteration)
- 9709/31 J25 —Q9 (numerical methods)
- 9709/32 J25 —Q6 (sign change / iteration)

Solutions

2.1 $f(1) = e - 3 = -0.282$, $f(2) = e^2 - 6 = 1.389$; opposite signs $\to$ a root lies between 1 and 2.

2.2 e.g. $x = \sqrt[3]{5 - x}$.

3.1 **B**. Convergence means the iterates settle towards a fixed value.

3.2 (a) $f(x) = x^3 - 4x - 1$: $f(2) = -1 < 0$, $f(2.2) = 10.648 - 8.8 - 1 = 0.848 > 0$; sign change $\to$ root between.
(b) $x^3 = 4x + 1 \Rightarrow x = \sqrt[3]{4x+1}$; $x_1 = \sqrt[3]{9.4} = 2.110$, $x_2 = 2.113$, $x_3 = 2.114$; $\alpha = 2.11$.

3.3 the decreasing curve $y = e^{-x}$ meets the line $y = x$ exactly **once**, so there is 1 root; $g(x) = e^{-x} - x$ has $g(0.5) = 0.607 - 0.5 > 0$ and $g(1) = 0.368 - 1 < 0$, so the root lies between 0.5 and 1.