A-LEVEL MATHEMATICS • EXERCISE SHEET

3.5 Integration (by parts, substitution, partial fractions)

Name: _____ Class: _____ Date: _____ **Total: 16 marks**

Objective

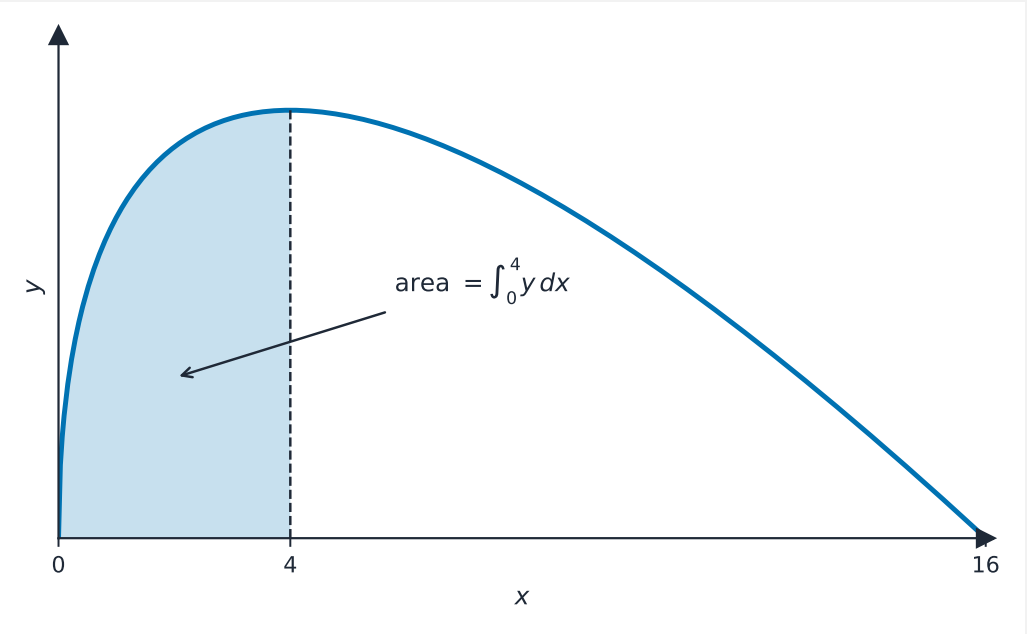
Build the skills to answer exam questions on **integration** 积分 in Pure 3 —the standard integral $\dfrac{1}{x^2+a^2}$, **integration by parts** 分部积分, **substitution** 换元积分, partial fractions, and the $\dfrac{kf'(x)}{f(x)}$ pattern.

You must be able to:

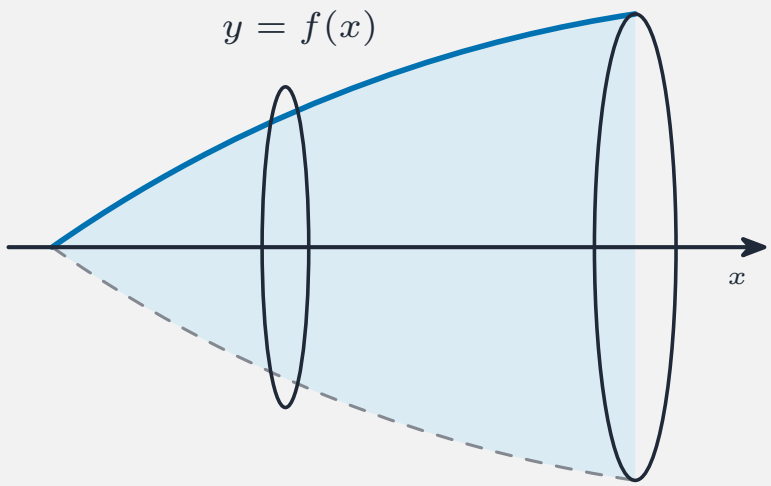
- recognise $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$ and $\int \frac{kf'(x)}{f(x)}\,dx = k\ln|f(x)| + C$
- integrate by parts and by a given substitution
- integrate a rational function via partial fractions

1 Worked examples

■ By parts & the log pattern



A definite integral is still an area; Pure 3 adds the methods to evaluate harder ones



rotate the region around the x-axis to sweep a solid

An application of integration: a volume of revolution

$\int x\cos x\,dx$ —by parts $(u = x,\ dv = \cos x\,dx)$: $= x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$.

$\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + C$ (top is the derivative of the bottom).

2 Practice

2.1 Find $\int \frac{1}{x^2+9}\,dx$. [2]

2.2 Find $\int \frac{3x^2}{x^3+1}\,dx$. [2]

3 Exam-style questions

3.1 $\int xe^x\,dx$ equals: [1]

- **A** $xe^x - e^x + C$
- **B** $xe^x + e^x + C$
- **C** $\frac{1}{2}x^2e^x + C$
- **D** $e^x + C$

3.2 Use the substitution $u = 1 + x^2$ to find $\displaystyle\int_0^2 x\sqrt{1+x^2}\,dx$. [5]

3.3 (a) Express $\dfrac{4}{(x)(x+2)}$ in partial fractions. [3]

(b) Hence find $\displaystyle\int_1^3 \frac{4}{x(x+2)}\,dx$, giving an exact answer. [3]

4 Go further

You are now ready for the real exam questions on this subtopic. Open the **3.5 Integration** past-paper sheet in the Library, or try these in **Practice mode**:

- 9709/31 N25 —Q1 (integration)
- 9709/31 J25 —Q9 (by parts / substitution)
- 9709/31 J25 —Q11 (integration)

Solutions

2.1 $\frac{1}{3}\tan^{-1}\dfrac{x}{3}+C$.

2.2 $\ln|x^3+1|+C$.

3.1 **A**. By parts: $xe^x-\int e^x\,dx = xe^x-e^x+C$.

3.2 $u=1+x^2 \Rightarrow du=2x\,dx$, so $x\,dx=\frac{1}{2}du$; limits $x=0\to u=1$, $x=2\to u=5$; $\displaystyle\int_1^5 \tfrac{1}{2}\sqrt{u}\,du = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_1^5 = \tfrac{1}{3}(5^{3/2}-1)$; $=\tfrac{1}{3}(11.180-1)=3.39$.

3.3 (a) $4=A(x+2)+Bx$; $x=0\Rightarrow A=2$; $x=-2\Rightarrow B=-2$; $\dfrac{2}{x}-\dfrac{2}{x+2}$.

(b) $\displaystyle\int_1^3\left(\frac{2}{x}-\frac{2}{x+2}\right)dx = \Big[2\ln x-2\ln(x+2)\Big]_1^3 = 2\ln\frac{x}{x+2}\Big|_1^3$; $=2\left(\ln\tfrac{3}{5}-\ln\tfrac{1}{3}\right)=2\ln\tfrac{9}{5}$.