A-LEVEL MATHEMATICS • EXERCISE SHEET

2.6 Numerical solution of equations

Name: ____________ Class: ________ Date: __________ **Total: 12 marks**

Objective

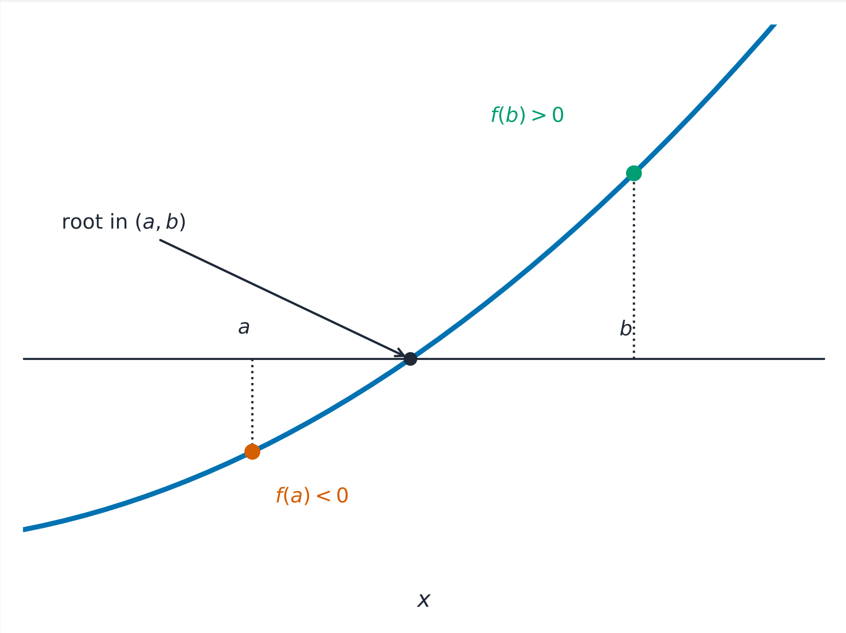
Build the skills to answer exam questions on the **numerical solution of equations** 方程的数值解—locating a **root** 根 by a **sign change** 变号, and finding it by **iteration** 迭代.

You must be able to:

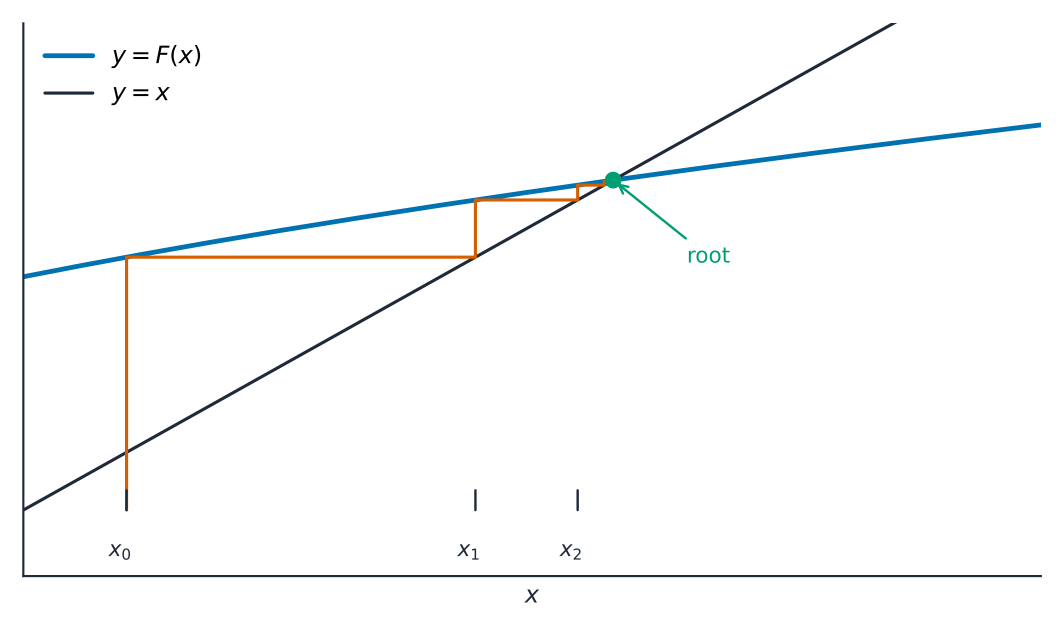
- use a sign change to show a root lies in an interval
- carry out an iteration $x_{n+1} = F(x_n)$ until it converges
- show a given equation rearranges into a stated iterative form

1 Worked examples

■ Sign change & iteration



If $f(a)$ and $f(b)$ have opposite signs, a root is trapped between a and b



Each step: up to $y = F(x)$, across to $y = x$; the staircase converges to the root

Iterate $x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ **from** $x_0 = -1.2$**:** $x_1 = -1.281$, $x_2 = -1.247$, $\cdots \to \beta = -1.26$ (3 s.f.).

2 Practice

2.1 $f(x) = x^3 - 5x + 1$. Evaluate $f(0)$ and $f(1)$, and state what they tell you. [2]

2.2 State the iterative formula you would use, written as $x_{n+1} = F(x_n)$, for the equation $x = \cos x$. [1]

3 Exam-style questions

3.1 A root of $f(x) = 0$ lies between $x = 2$ and $x = 3$ if: [1]

- **A** $f(2)$ and $f(3)$ are both positive
- **B** $f(2)$ and $f(3)$ have opposite signs
- **C** $f(2) = f(3)$
- **D** $f'(2) = 0$

3.2 The equation $x^3 - 6x + 2 = 0$ has a root α between 0 and 1.

(a) Verify this by a sign change. [2]

(b) Show that the equation can be written as $x = \dfrac{x^3 + 2}{6}$, and use the iteration $x_{n+1} = \dfrac{x_n^3 + 2}{6}$ with $x_0 = 0.3$ to find α to 2 d.p. [4]

3.3 Explain why a sign change between a and b does **not** guarantee exactly one root in that interval. [2]

4 Go further

You are now ready for the real exam questions on this subtopic. Open the **2.6 Numerical solution of equations** past-paper sheet in the Library, or try these in **Practice mode**:

- 9709/21 N25 —Q7 (iteration)
- 9709/22 N25 —Q6 (sign change / iteration)
- 9709/21 J25 —Q3 (numerical methods)

Solutions

2.1 $f(0) = 1$, $f(1) = 1 - 5 + 1 = -3$; opposite signs, so a root lies between 0 and 1.

2.2 $x_{n+1} = \cos x_n$.

3.1 B. Opposite signs (with a continuous graph) trap a root between them.

3.2 (a) $f(0) = 2 > 0$, $f(1) = 1 - 6 + 2 = -3 < 0$; opposite signs $\to$ root between 0 and 1.
(b) $x^3 - 6x + 2 = 0 \Rightarrow 6x = x^3 + 2 \Rightarrow x = \dfrac{x^3 + 2}{6}$; $x_1 = \dfrac{0.027 + 2}{6} = 0.3378$, $x_2 = 0.3397$, $x_3 = 0.3398$; $\alpha = 0.34$.

3.3 there could be several roots between a and b —the graph may cross the axis more than once (an odd number of times).