A-LEVEL MATHEMATICS • EXERCISE SHEET

2.5 Integration (standard forms, trapezium rule)

Name: ____________ Class: ________ Date: __________ **Total: 34 marks**

KEY WORDS trapezium rule 梯形法则 • integration 积分 • modulus 绝对值

Objective

Build the skills to answer exam questions on **integration 积分** — integrating e^{ax+b}, $\dfrac{1}{ax+b}$ and trig functions, powers of sin/cos via identities, and the **trapezium rule 梯形法则**.

You must be able to:

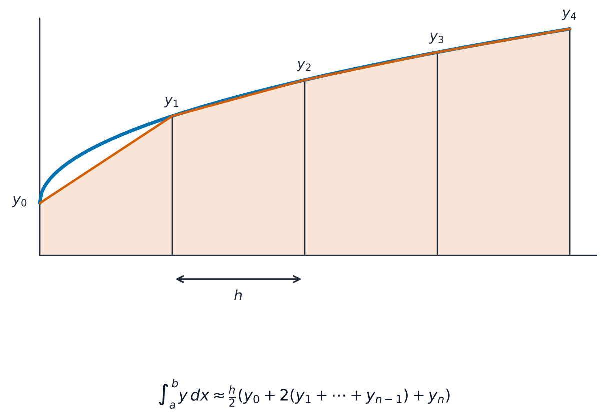
- integrate e^{ax+b}, $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$, $\sec^2(ax+b)$
- integrate $\sin^2/\cos^2$ using a double-angle identity
- estimate a definite integral with the trapezium rule

1 Worked examples

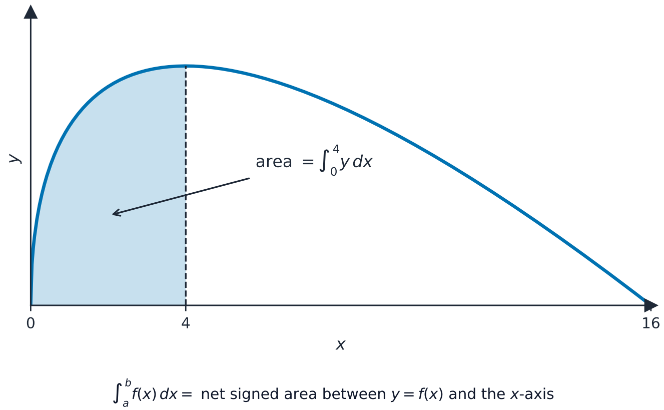
■ Powers of sin & the trapezium rule

$\int 6\sin^2 x\,dx$ — use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$:

$$\int (3 - 3\cos 2x)\,dx = 3x - \tfrac{3}{2}\sin 2x + C.$$



Each strip is a trapezium; their areas estimate $\int_a^b y\,dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})]$



The trapezium rule estimates this area under the curve

■ Choosing the method, and the standard results

Look at the integrand and pick the first method that fits:

1. **A standard form**, with a **linear** function of x inside. Integrate as usual and **divide by the coefficient of** x:

$$\int e^{ax+b}\,dx = \tfrac{1}{a}e^{ax+b} + C \qquad \int \frac{1}{ax+b}\,dx = \tfrac{1}{a}\ln|ax+b| + C$$

$$\int \sin(ax+b)\,dx = -\tfrac{1}{a}\cos(ax+b)+C \qquad \int \cos(ax+b)\,dx = \tfrac{1}{a}\sin(ax+b)+C$$

2. **A fraction whose numerator is the derivative of its denominator** $\to$ ln:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$$

3. **A substitution**, when a function and (a multiple of) its derivative both appear. Change the **limits** too if the integral is definite, and there is then no need to substitute back.
4. **Integration by parts**, for a product: $\int u\,\frac{dv}{dx}\,dx = uv - \int v\,\frac{du}{dx}\,dx$. Choose u as the factor that gets **simpler** when differentiated — a power of x, or $\ln x$.
5. **Partial fractions**, when the integrand is a proper algebraic fraction with a factorisable denominator.
6. **A trigonometric identity first**, when a power of a trig function appears: $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ turns an unintegrable square into two standard forms.

Worked example (parts, and why the choice of u matters). Find $\int x\,e^{2x}\,dx$.

1. Take $u = x$ (it differentiates to 1) and $\frac{dv}{dx} = e^{2x}$, so $v = \frac{1}{2}e^{2x}$.
2. $\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$.
3. $= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.

Choosing $u = e^{2x}$ instead gives an integral **harder** than the one you started with — that is the check that you picked the wrong way round.

△ **Every indefinite integral needs** $+C$, and every ln needs the **modulus**. Both are marks, and both are dropped more often than any actual method.

2 Practice

2.1 Find $\int e^{2x+1}\,dx$. [2]

2.2 Find $\int \frac{1}{2x+3}\,dx$. [2]

2.3 Find $\int \frac{6x}{3x^2+1}\,dx$. [2]

2.4 Find $\int \sin(3x-1)\,dx$. [2]

2.5 Use the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ to find $\int \cos^2 x\,dx$. [3]

3 Exam-style questions

3.1 $\int \cos 4x\,dx$ equals: [1]

- **A** $4\sin 4x + C$
- **B** $\frac{1}{4}\sin 4x + C$
- **C** $-\frac{1}{4}\sin 4x + C$
- **D** $-4\sin 4x + C$

3.2 Find $\int_0^{\pi/4} 4\cos^2 x\,dx$, giving an exact answer. [4]

3.3 Use the trapezium rule with **four** strips to estimate $\int_0^2 \sqrt{1+x^3}\,dx$, giving your answer to 3 s.f. [4]

3.4 (a) Find $\int x \ln x \, dx$, stating which factor you take as u and why. [4]

(b) Use the substitution $u = 1 + x^2$ to evaluate $\int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx$, giving your answer in exact form. [5]

(c) Express $\frac{5x+1}{(x+2)(x-1)}$ in partial fractions, and hence find $\int \frac{5x+1}{(x+2)(x-1)}\,dx$. [5]

Solutions

2.1 $\frac{1}{2}e^{2x+1} + C$.

2.2 $\frac{1}{2}\ln|2x+3| + C$.

2.3 The numerator is the derivative of the denominator; $= \ln|3x^2+1| + C$.

2.4 $-\frac{1}{3}\cos(3x-1) + C$.

2.5 $\int \frac{1}{2}(1+\cos 2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$.

3.1 **B**. $\int \cos 4x\,dx = \frac{1}{4}\sin 4x + C$.

3.2 $4\cos^2 x = 2(1+\cos 2x)$; $\int_0^{\pi/4}(2+2\cos 2x)\,dx = [2x+\sin 2x]_0^{\pi/4}$; $= \left(\frac{\pi}{2}+1\right) - 0 = \frac{\pi}{2}+1$.

3.3 $h = 0.5$; y at $x = 0, 0.5, 1, 1.5, 2$: 1, $\sqrt{1.125} = 1.0607$, $\sqrt{2} = 1.4142$, $\sqrt{4.375} = 2.0917$, $\sqrt{9} = 3$; $\int \approx \frac{0.5}{2}[1 + 3 + 2(1.0607 + 1.4142 + 2.0917)]$; $= 0.25[4 + 2(4.5666)] = 0.25(13.133) = 3.28$ (3 s.f.).

3.4 (a) Take $u = \ln x$, because it differentiates to the simpler $\frac{1}{x}$, and $\dfrac{dv}{dx} = x$ so $v = \frac{1}{2}x^2$; $= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx$; $= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$.

(b) $u = 1 + x^2$ so $\dfrac{du}{dx} = 2x$ and $x\,dx = \frac{1}{2}\,du$; the limits become $u = 1$ and $u = 5$; $\int_1^5 \frac{1}{2}u^{-1/2}\,du = \left[u^{1/2}\right]_1^5$; $= \sqrt{5} - 1$.

(c) $\dfrac{5x+1}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$, so $5x+1 = A(x-1) + B(x+2)$; $x = 1$ gives $B = 2$, $x = -2$ gives $A = 3$; $\int \dfrac{3}{x+2} + \dfrac{2}{x-1}\,dx$; $= 3\ln|x+2| + 2\ln|x-1| + C$.