A-LEVEL MATHEMATICS • EXERCISE SHEET

1.7 Differentiation

Name: ____________ Class: ________ Date: __________ **Total: 16 marks**

KEY WORDS stationary point 驻点 • normal 法线 • chain rule 链式法则 • tangents and normals 切线与法线 • second derivative 二阶导数

Objective

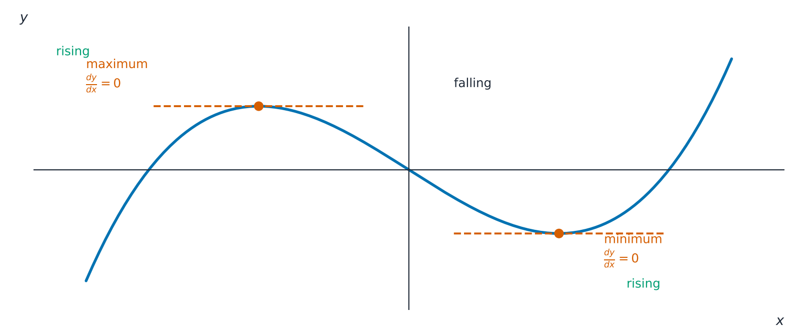
Build the skills to answer exam questions on **differentiation 微分** — the power and **chain rule 链式法则**, **tangents and normals 切线与法线**, increasing/decreasing, and **stationary points 驻点**.

You must be able to:

- differentiate powers and use the chain rule
- find the equations of a tangent and a normal
- find and classify stationary points (max/min) with $f''(x)$

1 Worked examples

■ Stationary points



stationary points: $\frac{dy}{dx} = 0$ · use $\frac{d^2y}{dx^2}$ or first-derivative sign to classify

$$\textit{At a stationary point } \frac{dy}{dx} = 0;\ f''(x) > 0 \to \textit{minimum},\ f''(x) < 0 \to \textit{maximum}$$

$y = 4x^{1/2} - x$ **has a maximum at** $x = a$**. Find** a**.**

$$\frac{dy}{dx} = 2x^{-1/2} - 1 = 0 \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4,\ \text{so } a = 4.$$

Chain rule: $\frac{d}{dx}(2x-1)^4 = 4(2x-1)^3 \cdot 2 = 8(2x-1)^3$.

2 Practice

2.1 Differentiate $y = 3x^4 - \frac{2}{x}$. [2]

2.2 Find $\frac{dy}{dx}$ when $y = (3x+1)^5$. [2]

3 Exam-style questions

3.1 The gradient of $y = x^3 - 2x$ at $x = 2$ is: [1]

- **A** 4
- **B** 6
- **C** 10
- **D** 12

3.2 A curve has equation $y = x^3 - 3x^2 - 9x + 5$.

(a) Find the coordinates of the two stationary points. [4]

(b) Use the second derivative to determine the nature of each, and hence write down the set of values of x for which the curve is decreasing. [3]

3.3 The curve $y = \dfrac{8}{x}$ passes through $P(2, 4)$. Find the equation of the **normal** to the curve at P. [4]

Solutions

2.1 $\dfrac{dy}{dx} = 12x^3 + \dfrac{2}{x^2}$.

2.2 $\dfrac{dy}{dx} = 5(3x+1)^4 \cdot 3 = 15(3x+1)^4$.

3.1 C. $\dfrac{dy}{dx} = 3x^2 - 2$; at $x = 2$, $= 12 - 2 = 10$.

3.2 (a) $\dfrac{dy}{dx} = 3x^2 - 6x - 9 = 3(x-3)(x+1) = 0$; $x = 3$ or $x = -1$; $y(3) = 27 - 27 - 27 + 5 = -22$ and $y(-1) = -1 - 3 + 9 + 5 = 10$, so $(3, -22)$ and $(-1, 10)$.
(b) $\dfrac{d^2y}{dx^2} = 6x - 6$; at $x = 3$ it is $12 > 0 \to$ minimum, at $x = -1$ it is $-12 < 0 \to$ maximum; the curve falls between the maximum and the minimum, so it is decreasing for $-1 < x < 3$.

3.3 $\dfrac{dy}{dx} = -\dfrac{8}{x^2}$; at $x = 2$, gradient $= -2$; normal gradient $= \frac{1}{2}$; $y - 4 = \frac{1}{2}(x - 2) \Rightarrow y = \frac{1}{2}x + 3$.