Exercise walk-through

Continuous random variables ▪ 6.3

eXercise

You must be able to

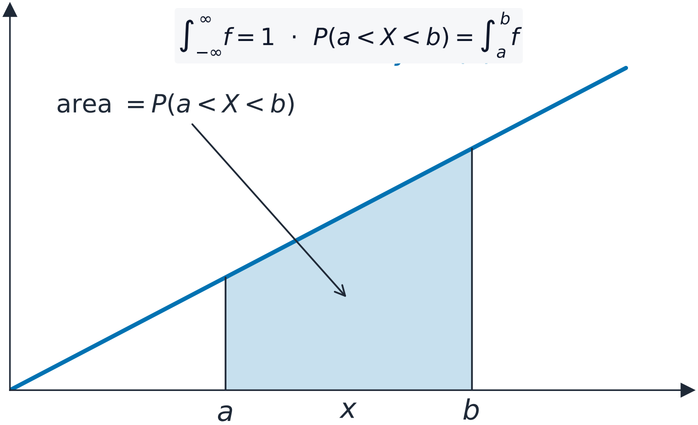
- use $\int f(x)\,dx = 1$ to find a constant; find $P(a < X < b) = \int_a^b f\,dx$
- find $E(X) = \int xf(x)\,dx$ (and $\mathrm{Var}(X)$)
- find $F(x)$ and use it for the median/percentiles

Method — Density and expectation

$f(x) = \frac{1}{2}x$ **for** $0 \leqslant x \leqslant 2$**:** $E(X) = \displaystyle\int_0^2 x \cdot \tfrac{1}{2}x\,dx = \left[\tfrac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$.

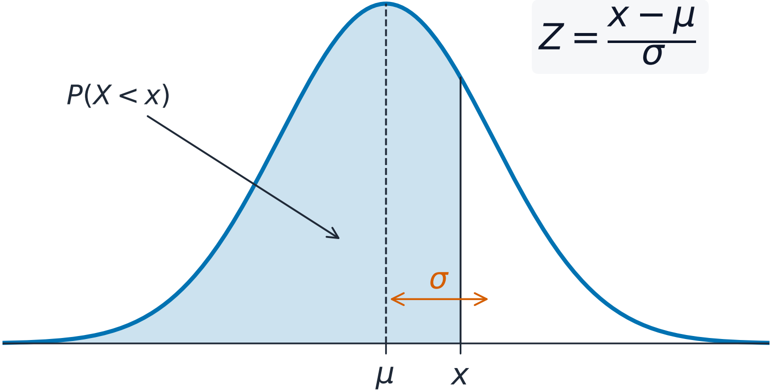
Median m**:** solve $F(m) = 0.5$.

Method — Density and expectation



A density curve with the region a to b shaded

Method — Density and expectation



$$Z = \frac{x - \mu}{\sigma}$$

The normal distribution is a continuous random variable

Practice 2.1 ▪ 2 marks

$f(x) = kx$ for $0 \leqslant x \leqslant 2$. Show that $k = \frac{1}{2}$.

Solution 2.1

Method: Density and expectation

Worked steps

$$\int_0^2 kx\,dx = \left[\frac{kx^2}{2}\right]_0^2 = 2k = 1 \Rightarrow k = \frac{1}{2}$$ **M1** **A1**.

Practice 2.2 ▪ 2 marks

For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $P(X < 1)$.

Solution 2.2

Worked steps

$$\int_0^1 \frac{1}{2}x\,dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4} \quad \textbf{M1}\,\textbf{A1}.$$

Exam-style 3.1 ▪ 1 mark

For a valid pdf, $\int_{-\infty}^{\infty} f(x)\, dx$ must equal:

A 0

B 0.5

C 1

D ∞

Solution 3.1

A 0

B 0.5

C 1 ✓

D ∞

Worked steps

C. The total area under a pdf is 1.

Exam-style 3.2 ▪ 3 marks

A continuous random variable has $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \leqslant x \leqslant 1$ (and 0 elsewhere).

(a) Verify that $\int_{-1}^{1} f(x)\, dx = 1$.

(b) Find $P(X > 0.5)$.

Solution 3.2

Method: Density and expectation

Worked steps

(a) $\int_{-1}^{1} \frac{3}{4}(1 - x^2)\, dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{3}{4}\left(\frac{2}{3} - (-\frac{2}{3})\right) = \frac{3}{4} \cdot \frac{4}{3} = 1$ **M1** **M1** **A1**.

(b) $\int_{0.5}^{1} \frac{3}{4}(1 - x^2)\, dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{0.5}^{1} = \frac{3}{4}\left(\frac{2}{3} - (0.5 - 0.0417)\right) = \frac{3}{4}(0.6667 - 0.4583) = 0.156$ **M1** **M1** **A1**.

Exam-style 3.3 ▪ 3 marks

A variable has $f(x) = \frac{3}{8}x^2$ for $0 \leqslant x \leqslant 2$.
(a) Find $E(X)$.
(b) Find the median m.

Solution 3.3

Method: Density and expectation

Worked steps

(a) $E(X) = \int_0^2 x \cdot \frac{3}{8}x^2 \, dx = \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2 = \frac{3}{8} \cdot 4 = \frac{3}{2}$ M1 M1 A1.

(b) $F(m) = \int_0^m \frac{3}{8}x^2 \, dx = \frac{m^3}{8} = 0.5 \Rightarrow m^3 = 4 \Rightarrow m = 1.59$ M1 M1 A1.