# Past-paper walk-through

## The Poisson distribution ▪ 6.1

eXercise

# Questions in this set

- 9709/61 N25 Q1 ▪ 9 marks
- 9709/61 N25 Q3 ▪ 6 marks
- 9709/62 N25 Q1 ▪ 8 marks
- 9709/61 J25 Q1 ▪ 3 marks
- 9709/61 J25 Q5 ▪ 11 marks

# Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

# Question 1

9709/61 N25 ▪ Q1 ▪ 9 marks

The random variables $X$ and $Y$ have independent distributions $X \sim \mathrm{Po}(3)$ and $Y \sim \mathrm{Po}(2)$ respectively.

**(a)** Find $P(2 < X < 5)$. **[2]**

**(b)** Find $P(X + Y > 2)$. **[3]**

**(c)** The total of 100 random values of $X$ and 150 random values of $Y$ is denoted by $T$. Use a suitable approximating distribution to find $P(T < 560)$. **[4]**

# Solution 1

## (a)

**Mark scheme**

- $P(2 < X < 5) = P(X = 3) + P(X = 4) = e^{-3}\left(\frac{27}{6} + \frac{81}{24}\right) = 0.392.$

# Solution 1

## (b) Worked steps

Independent Poisson variables **add**, and their means add with them:

$$X \sim \mathrm{Po}(\lambda_1),\ \ Y \sim \mathrm{Po}(\lambda_2)\ \Rightarrow\ X + Y \sim \mathrm{Po}(\lambda_1 + \lambda_2)$$

Here $X + Y \sim \mathrm{Po}(5)$.

- $P(X + Y > 2) = 1 - P(X + Y \leq 2)$
- $= 1 - e^{-5}\left(1 + 5 + \frac{5^2}{2}\right) = 1 - e^{-5}(18.5) = \mathbf{0.875}$

# Solution 1

Two habits. **Use the complement** — "more than 2" is three terms subtracted from 1, where the direct route is an infinite sum. And read the inequality carefully: $> 2$ excludes 2, so $P(X \leq 2)$ is the right complement; $\geq 2$ would need $P(X \leq 1)$. The additive property is only true for **independent** variables, and it is worth saying so: this is the one distribution where the sum stays in the same family.

## Mark scheme

- $X + Y \sim \mathrm{Po}(5)$: $P(X + Y > 2) = 1 - e^{-5}(1 + 5 + 12.5) = 0.875.$

# Solution 1

## (c) Worked steps

A Poisson with a large mean is approximately **normal**, with the variance equal to the mean.

- $E(T) = 100(3) + 150(2) = 600$
- $\mathrm{Var}(T) = 600$ — for a Poisson the variance **equals** the mean, and both add.
- $T \approx N(600, 600)$

Now the probability, with a **continuity correction**:

# Solution 1

- $P(T < 560) \approx P\left(Z < \frac{559.5 - 600}{\sqrt{600}}\right) = P(Z < -1.653) = \mathbf{0.0492}$

Three marks-worth of detail:

- **State the approximating distribution** with both parameters — the question asks for a suitable one, so naming $N(600, 600)$ is part of the answer.
- **Continuity correction**: $T < 560$ on a discrete variable means $T \leq 559$, so the boundary is **559.5**. Using 560 is the standard error and shifts the answer.
- **Divide by the standard deviation,** $\sqrt{600}$, not by the variance.

# Solution 1

The approximation is justified because $\lambda = 600$ is large; a rule of thumb is $\lambda > 15$.

**Mark scheme**

- $E(T) = 100(3) + 150(2) = 600$, $\mathrm{Var}(T) = 600$, so $T \approx N(600, 600)$.
  $P(T < 560) \approx P\left(Z < \frac{559.5-600}{\sqrt{600}}\right) = P(Z < -1.653) = 0.0492.$

# Question 3

9709/61 N25 ▪ Q3 ▪ 6 marks

The data produced by a certain data entry firm include a small number of incorrect characters that occur at random, with proportion $p$; experience shows $p = 0.0001$. A particular data set contains 14500 characters, of which $X$ are incorrect.

**(a)** Use a suitable approximating distribution to find $P(X < 4)$. **[3]**

**(b)** The management wishes to decrease $p$ by training so that, for a data set of 14500 characters, $P(X = 0)$ for the new $p$ is double the value of $P(X = 0)$ when $p = 0.0001$. Use a suitable approximating distribution to find the new value of $p$. **[3]**

# Solution 3

## (a) Worked steps

A binomial with a **large** $n$ **and a small** $p$ is approximated by a Poisson whose mean is $np$.

- $\lambda = 14500 \times 0.0001 = 1.45$, so $X \approx \mathrm{Po}(1.45)$.
- $P(X < 4) = P(X \leq 3)$ — four terms, since $X$ is a whole number:
- $= e^{-1.45}\left(1 + 1.45 + \frac{1.45^2}{2} + \frac{1.45^3}{6}\right) = \mathbf{0.940}$

# Solution 3

Three points:

- **State the distribution and its parameter.** "Use a suitable approximating distribution" is an instruction to name $\mathrm{Po}(1.45)$, and that is a mark.
- $X < 4$ **means** $X \leq 3$, so the sum runs $k = 0, 1, 2, 3$ — four terms, not three. This is the commonest slip in Poisson questions and it is purely a reading error.
- **Justify the approximation** if there is room: $n$ is large and $p$ is small, so $np$ is moderate — the conditions the Poisson approximation needs.

# Solution 3

No continuity correction here: both the binomial and the Poisson are **discrete**, so nothing is being smoothed. The correction belongs only to a **normal** approximation.

**Mark scheme**

- $X \approx \mathrm{Po}(14500 \times 0.0001) = \mathrm{Po}(1.45)$.
  $P(X < 4) = e^{-1.45}\left(1 + 1.45 + \frac{1.45^2}{2} + \frac{1.45^3}{6}\right) = 0.940.$

# Solution 3

**(b)**

**Mark scheme**

- $P(X=0)=e^{-14500p}$. Require $e^{-14500p}=2e^{-1.45}\Rightarrow 14500p=1.45-\ln 2=0.757\Rightarrow p=5.22\times 10^{-5}$.

# Question 1

9709/62 N25 ▪ Q1 ▪ 8 marks

The number, $X$, of used computers donated to a charity has a constant average rate of 2.4 computers per week.

**(a)** State a necessary condition for $X$ to have a Poisson distribution. **[1]**

**(b)** Now assume that $X$ has a Poisson distribution. Calculate the probability that the number of computers donated during a 4-week period is more than 6 and less than 9. **[3]**

**(c)** Use a suitable approximating distribution to calculate the probability that more than 50 computers are donated during a 20-week period. **[4]**

# Solution 1

## (a)

**Mark scheme**

- The computers are donated independently of one another (singly and at random).

# Solution 1

## (b) Worked steps

The Poisson mean **scales with the interval**, so multiply it by the number of periods first.

- Four-week mean: $\lambda = 4 \times 2.4 = 9.6$
- $P(6 < X < 9)$ is a **strict** double inequality, so it means $X = 7$ or $X = 8$ — two terms only.
- $= e^{-9.6}\left(\dfrac{9.6^7}{7!} + \dfrac{9.6^8}{8!}\right) = \mathbf{0.222}$

# Solution 1

The reading is the whole difficulty. “More than 6 and less than 9” excludes both 6 and 9, so the values are 7 and 8. Including either endpoint changes the answer materially, and it is the error the question is built to catch.

# Solution 1

Scale the mean **before** doing anything else. Computing with the weekly mean and multiplying the probability afterwards is not equivalent and gives a quite different number.

**Mark scheme**

- 4-week mean $\lambda = 9.6$.
  $P(6 < X < 9) = P(X = 7) + P(X = 8) = e^{-9.6}\left(\frac{9.6^7}{7!} + \frac{9.6^8}{8!}\right) = 0.222$.

# Solution 1

## (c) Worked steps

Twenty weeks makes the mean large enough for a **normal approximation**, with variance equal to the mean.

- $\lambda = 20 \times 2.4 = 48$, so $X \approx N(48, 48)$.
- With a continuity correction, "more than 50" means $X \geq 51$, so the boundary is **50.5**:
- $P(X > 50) \approx P\left(Z > \frac{50.5 - 48}{\sqrt{48}}\right) = P(Z > 0.361) = \mathbf{0.359}$

# Solution 1

Three things carry it: **name the approximating distribution** with both parameters; use the **continuity correction** at 50.5 (a discrete "more than 50" starts at 51, so the midpoint is 50.5); and divide by $\sqrt{48}$, the standard deviation, not by 48.
Sense-check the direction: 50 is just above the mean of 48, so the probability should be a little under a half. 0.359 is consistent.

**Mark scheme**

- 20-week mean $\lambda = 48$, approximated by $N(48, 48)$.
  $P(X > 50) \approx P\left(Z > \frac{50.5-48}{\sqrt{48}}\right) = P(Z > 0.361) = 0.359.$

# Question 1

9709/61 J25 ▪ Q1 ▪ 3 marks

It is known that 1

# Solution 1

## Worked steps

Two things settle every Poisson question, and both come before any arithmetic.
**1. Scale the mean to the interval the question asks about.** The Poisson parameter is a rate times a duration, so a mean of $\lambda$ per week is $3\lambda$ over three weeks. Compute the new $\lambda$ first and write it down.
**2. Translate the wording into a list of values**, remembering $X$ is a whole number:

- "at least $k$" $\rightarrow X \geq k$, so use $1 - P(X \leq k-1)$
- "more than $k$" $\rightarrow X \geq k+1$, so use $1 - P(X \leq k)$
- "at most $k$" $\rightarrow X \leq k$
- "fewer than $k$" $\rightarrow X \leq k-1$
- "more than $a$ and fewer than $b$" $\rightarrow$ the values strictly between, so $a+1$ up to $b-1$

## Solution 1

Then apply $P(X = r) = e^{-\lambda}\dfrac{\lambda^r}{r!}$, summing the terms you listed.
**Use the complement whenever the tail is the long one.** "At least 2" is $1 - P(0) - P(1)$, two terms; computed directly it is an infinite sum.

# Solution 1

Two habits that protect the marks: **write the parameter you are using** — $X \sim \mathrm{Po}(\lambda)$ with the number in it — and **list the values before summing**, because almost every lost mark in this topic is a boundary read one out.

## Mark scheme

- $X \sim B(400, 0.01) \approx \mathrm{Po}(4)$. $\mathrm{P}(X \leqslant 3) = e^{-4}\left(1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!}\right) = 0.433$.

# Question 5

9709/61 J25 ▪ Q5 ▪ 11 marks

**(a)(i)** Find $\mathrm{P}(3 \leqslant W + X \leqslant 5)$. **[2]**

**(a)(ii)** The random variable $S$ is the sum of 100 independent values of $W$ and 200 independent values of $X$. Use a suitable approximation to find $\mathrm{P}(S > 600)$. **[6]**

**(b)** The random variable $Y$ has the distribution $\mathrm{Po}(\lambda)$, where $\lambda > 0$. It is given that $\frac{5}{2}\mathrm{P}(Y = 3) + \mathrm{P}(Y = 4) = \mathrm{P}(Y = 5)$. Find the value of $\lambda$. **[3]**

# Solution 5

## (a)(i)

**Mark scheme**

- $W + X \sim \mathrm{Po}(3.5)$. $\mathrm{P}(3 \leqslant W + X \leqslant 5) = e^{-3.5}\left(\frac{3.5^3}{3!} + \frac{3.5^4}{4!} + \frac{3.5^5}{5!}\right) = 0.537.$

# Solution 5

## (a)(ii)

**Mark scheme**

- $S \sim N(580, 580)$ (since $E(S) = \mathrm{Var}(S) = 100(1.2) + 200(2.3) = 580$). With continuity correction:
  $\mathrm{P}(S > 600) = 1 - \Phi\left(\frac{600.5-580}{\sqrt{580}}\right) = 1 - \Phi(0.851) = 0.197.$

# Solution 5

## (b)

### Mark scheme

- $\frac{5}{2} \cdot \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} = \frac{\lambda^5}{5!} \Rightarrow \lambda^2 - 5\lambda - 50 = 0 \Rightarrow (\lambda - 10)(\lambda + 5) = 0 \Rightarrow \lambda = 10.$