Exercise walk-through

Differential equations ▪ 3.8

eXercise

You must be able to

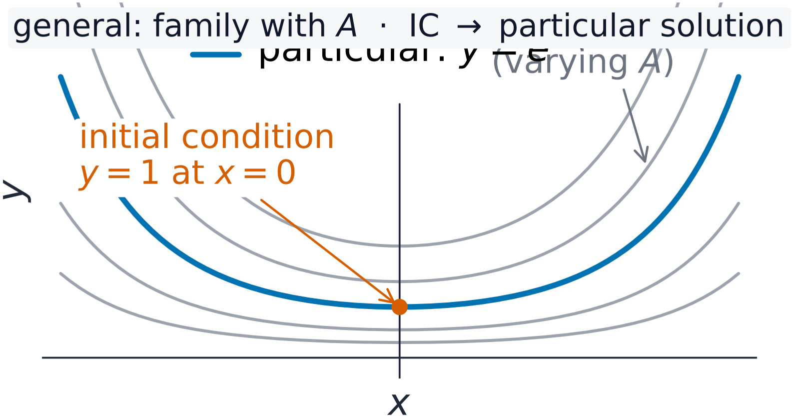
- separate variables and integrate both sides
- use an initial condition to find the constant
- form and solve a differential equation from a rate description

Method — Separate and solve

Solve $\dfrac{dy}{dx} = xy$**,** $y = 1$ **at** $x = 0$**.**

$$\int \frac{1}{y}\,dy = \int x\,dx \Rightarrow \ln y = \tfrac{1}{2}x^2 + c \Rightarrow y = Ae^{x^2/2}; \quad y(0) = 1 \Rightarrow A = 1, \text{ so } y = e^{x^2/2}.$$

Method — Separate and solve



A family of solution curves with one selected by an initial condition

Practice 2.1 ▪ 1 mark

Separate the variables for $\frac{dy}{dx} = \frac{y}{x}$ (write each side ready to integrate).

Solution 2.1

Method: Separate and solve

Worked steps

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx \quad \boxed{\mathbf{B1}}.$$

Practice 2.2 ▪ 2 marks

Solve $\dfrac{dy}{dx} = 3y$, giving the general solution.

Solution 2.2

Method: Separate and solve

Worked steps

$\ln y = 3x + c \Rightarrow y = Ae^{3x}$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

The general solution of $\dfrac{dy}{dx} = ky$ is:

A $y = kx + c$

B $y = Ae^{kx}$

C $y = Ax^k$

D $y = \frac{1}{2}kx^2 + c$

Solution 3.1

Method: Separate and solve

A $y = kx + c$

B $y = Ae^{kx}$ ✓

C $y = Ax^{k}$

D $y = \frac{1}{2}kx^{2} + c$

Worked steps

B. $\dfrac{dy}{dx} = ky$ gives $y = Ae^{kx}$.

Exam-style 3.2 ▪ 5 marks

Solve $\dfrac{dy}{dx} = 2x(y+1)$, given $y = 0$ when $x = 0$. Give y explicitly.

Solution 3.2

Method: Separate and solve

Worked steps

$\int \frac{1}{y+1}\,dy = \int 2x\,dx$ **M1**; $\ln(y+1) = x^2 + c$ **M1**; $y(0) = 0 \Rightarrow c = 0$ **M1**; $y + 1 = e^{x^2}$ **M1**; $y = e^{x^2} - 1$ **A1**.

Exam-style 3.3 ▪ 4 marks

A tank loses water so that the depth h falls at a rate proportional to $\sqrt{h}$: $\dfrac{dh}{dt} = -k\sqrt{h}$.
Initially $h = H$.
(a) Solve to show $\sqrt{h} = \sqrt{H} - \frac{1}{2}kt$.
(b) Find, in terms of H and k, the time for the tank to empty.

Solution 3.3

Method: Separate and solve

Worked steps

(a) $\int h^{-1/2}\,dh = \int -k\,dt$ **M1**; $2\sqrt{h} = -kt + c$ **M1**; $h = H$ at $t = 0 \Rightarrow c = 2\sqrt{H}$ **M1**; $\sqrt{h} = \sqrt{H} - \frac{1}{2}kt$ **A1**.

(b) empty when $h = 0$: $0 = \sqrt{H} - \frac{1}{2}kt \Rightarrow t = \dfrac{2\sqrt{H}}{k}$ **M1** **A1**.