Past-paper walk-through

Numerical solution of equations ▪ 3.6

eXercise

Questions in this set

- 9709/31 N25 Q9 ▪ 8 marks
- 9709/31 J25 Q9 ▪ 10 marks
- 9709/32 J25 Q6 ▪ 7 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 9

9709/31 N25 ▪ Q9 ▪ 8 marks

(a) By sketching a suitable pair of graphs, show that the equation $\sec 2x = -e^x$ has only one root in the interval $0 < x < \frac{1}{2}\pi$. **[2]**

(b) Show by calculation that this root lies between 0.9 and 1. **[2]**

(c) Show that if a sequence of values given by the iterative formula $x_{n+1} = \frac{1}{2}\cos^{-1}(-e^{-x_n})$ converges, then it converges to the root of the equation in part (a). **[1]**

(d) Use the iterative formula given in part (c) to calculate x correct to 3 decimal places. Give the result of each iteration to 5 decimal places. **[3]**

Solution 9

(a)

Mark scheme

- On $0 < x < \frac{1}{2}\pi$, $y = \sec 2x$ goes to $-\infty$ then up to -1 on $(\frac{\pi}{4}, \frac{\pi}{2})$, while $y = -e^x$ is negative and decreasing; the two graphs cross exactly once.

Solution 9

(b)

Mark scheme

- $f(x) = \sec 2x + e^x$: $f(0.9) = -1.94 < 0$ and $f(1) = 0.32 > 0$, so the root lies between 0.9 and 1.

Solution 9

(c)

Mark scheme

- If $x_{n+1} = \frac{1}{2}\cos^{-1}(-e^{-x_n}) \to L$, then $\cos 2L = -e^{-L}$, so $\sec 2L = -e^{L}$ — i.e. L satisfies $\sec 2x = -e^{x}$.

Solution 9

(d)

Mark scheme

- Iterating from $x_0 = 0.95$: 0.98417, 0.97667, 0.97823, $0.97791, \ldots \Rightarrow x = 0.978$ (3 d.p.).

Question 9

9709/31 J25 ▪ Q9 ▪ 10 marks

The constant a is such that $\displaystyle\int_1^a 6x\ln x\,dx = 4$.

(a) Show that $a = \exp\left(\frac{1}{6}\left(\frac{5}{a^2} + 3\right)\right)$, where $\exp(x)$ denotes e^x. **[5]**

(b) Verify by calculation that a lies between 2 and 2.1. **[2]**

(c) Use an iterative formula based on the equation in part (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. **[3]**

Solution 9

(a)

Mark scheme

- By parts: $\int 6x \ln x\, dx = 3x^2 \ln x - \frac{3}{2}x^2$. Limits 1 to a equal 4: $3a^2 \ln a - \frac{3}{2}a^2 + \frac{3}{2} = 4 \Rightarrow \ln a = \frac{1}{6}\left(\frac{5}{a^2} + 3\right)$, i.e. $a = \exp\left(\frac{1}{6}\left(\frac{5}{a^2} + 3\right)\right)$ (AG).

Solution 9

(b)

Mark scheme

- With $f(x) = 6x \ln x - 3x^2$: $f(2) = 4.636 < 5$ and $f(2.1) = 6.402 > 5$, so a lies between 2 and 2.1.

Solution 9

(c)

Mark scheme

- Iterating $a_{n+1} = \exp\left(\frac{1}{6}\left(\frac{5}{a_n^2} + 3\right)\right)$ converges to $a = 2.02$ (2 d.p.).

Question 6

9709/32 J25 ▪ Q6 ▪ 7 marks

(a) By sketching a suitable pair of graphs, show that the equation $|x-2| = 2\sin\frac{1}{2}x$ has only one root in the interval $0 < x < \pi$. **[2]**

(b) Show by calculation that this root lies between 1 and 1.5. **[2]**

(c) Use the iterative formula $x_{n+1} = 2 - 2\sin\frac{1}{2}x_n$ with an initial value of 1.03 to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. **[3]**

Solution 6

(a)

Mark scheme

- $y = |x - 2|$ is a V with vertex $(2, 0)$; $y = 2\sin\frac{1}{2}x$ rises from O to $(\pi, 2)$. The graphs meet exactly once in $0 < x < \pi$.

Solution 6

(b)

Mark scheme

- Let $f(x) = (2 - x) - 2\sin\frac{1}{2}x$. $f(1) = +0.041 > 0$ and $f(1.5) = -0.863 < 0$; the sign change shows the root lies between 1 and 1.5.

Solution 6

(c)

Mark scheme

- $x_1 = 1.0148$, $x_2 = 1.0282$, $x_3 = 1.0164$, $x_4 = 1.0268, \ldots \Rightarrow$ root $= 1.02$ (2 d.p.).